NUMERICAL MODELING OF TURBULENT GAS FLOW IN POROUS MEDIA: A FRACTIONAL DIFFUSION APPROACH

BY

Rami M. Alloush

A Thesis Presented to the

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In Partial Fulfillment of the

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In

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Dedication

To my beloved wife Amira, dear parents, brother and sister and all my amazing friends for their endless love and support ♥

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ABSTRACT

Full Name	:	Rami Mohamed Alloush
Thesis Title	:	Numerical Modeling of Turbulent Gas Flow in Porous Media: A Fractional Diffusion Approach
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Understanding the physics of fluid flow in a porous media is of high interest in many fields, such as water extraction from aquifers and the geo-mechanics related to soil mechanics. The oil and gas industry is no exception to this, and the topic is getting more and more attention with the increasing energy demand. Our current understanding of fluid flow in the porous media is based on years of research on this topic in the said fields, and the outcomes of the future work should provide the basis for better predictions and decisions. We aim in this research to explore the physics of fluid flow in porous media (such as in Oil and Gas Reservoirs) based upon the concept of anomalous diffusion cases; where classical Darcy's law and its modification for gas (Forchheimer's Equation) do not fully describe the fluid physics.

Henry Darcy was the first to develop an equation describing fluid flow through porous media. His equation was developed to calculate flow rate of water through sand beds. What we know now as permeability, was first used to estimate the conductivity of sand beds in his experiment. Darcy's Law has good analogy to Fick's Law in the diffusion theory, which describes the usual diffusion in porous media. However, the usual diffusion is not always the case, as there are several cases where the paths of the fluid flow are complex, or the velocity of the fluid is very high, hence, the anomalous diffusion takes place.

Fractional derivatives has been used as a mean to describe the anomalous diffusion process, this requires the modification of the conventional laws (Darcy's law for liquid and Forchheimer's for gas). In this work, we implement the application of the memory formalisms on the pressure flux term for gas flow, by modifying the Forchheimer's Equation. We use fractional order derivatives to represent the memory formalisms and its effect on the pressure distribution.

The modified Forchheimer's Equation is used to derive a diffusivity equation for the gas flow, and its solution is obtained numerically. The pressure behavior of the gas reservoir is modeled after incorporating the memory parameter (α), and the effect on the pressure distribution over time is analyzed.

The results of this study show that the bottom hole pressure is affected by the memory parameter and that the α affects the calculation of permeability values from graphical analysis. And from that, we can see that the pressure data obtained from normal diffusion models will be erroneous if the actual fluid was an anomalous flow.

As a validation strategy, the permeability (k) and (α) are estimated using non-linear regression (Levenberg-Marquardt algorithm) considering both normal and fractional diffusion to show the importance of the model modification on the parameters estimation process, and how ignoring the anomalous effect would result in less accurate results.

ملخص الرسالة

الاسم الكامل: رامي محمد علوش

عنوان الرسالة: النمذجة العددية لتدفق الغازات المضطربة في المواد المسامية: باستخدام منهج الإنتشار الجزئي

التخصص: هندسة البترول

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إن فهم الفيزياء وراء سير الموائع في الأوساط المسامية لذو أهمية بالغة في عدة مجالات، بدءاً من استخراج المياه من جوف الأرض وحتى علوم ميكانيا التربة. وبالطبع فإن صناعة الزيت والغاز ليست بمنأى عن ذلك، فالفهم الجيد لهذا الموضوع يزداد أهمية يوماً بعد يوم، خصوصاً مع الأرتفاع المتزايد للطلب على الطاقة.

إن فهمنا الحالي لسير الموائع في الأوساط المسامية يعتمد على سنوات طويلة من البحث والدراسة، والمزيد من الدراسات الحديثة يفترض أن تساعدنا على الوصول إلى فهم أعمق وبالتالي القدرة على اتخاذ قرارات أصحّ. إن الهدف الرئيسي من هذه الدراسة هو محاولة الوصول إلى فهم أعمق لقوانين الفيزياء اللتي تحكم سير الموائع في الأوساط المسامية (كما في حالة خزانات الزيت والغاز)، وذلك بالنظر إليها في ضوء حالات التدفق المضطرب؛ حيث تكون القوانين التقليديه (كقانون لعالم هنري دارسي) ومثيله الخاص بالغازات (معادلة فورتشايمر) غير صالحين لوصف فيزياء تلك الموائع بالدقة الملطوبة.

في هذه الدراسة، تم تعديل معادلة فورتشايمر لتكون صالحة لوصف السير المضطرب للغاز في الأوساط المسامية، وتم اختبار المعادلة بعد التعديل على أكثر من نموذج محاكاة للتأكد من أهمية التعديل وقدرته على حساب نتائج أفضل.

CHAPTER 1
INTRODUCTION

1.1 Introduction and Problem Description

Energy demand is known to be one the biggest issues of our societies. With the increasing demand every day and the continuous consumption of our existing resources, it is vital to find new resources and better manage the already existing ones. The extraction of oil and gas from petroleum reservoirs is the main energy resource upon which the world heavily depends, and as long as the contrast between world oil supply and demand is increasing, new methods will be required to make an efficient use of this resource.

The work in the oil and gas industry is divided into many different stages, all of them need to be well planned and executed, in order to be able to get the best out of any hydrocarbon reserves. There are many steps and approaches used to best manage a hydrocarbon reservoir and get the maximum recovery out of it. The term "Petroleum Reservoir Management" is usually used to describe these processes, and can be defined as the dynamic procedure used to recognize the uncertainties in the performance of any reservoir. It looks for any action that can reduce the uncertainties, while optimizing the reservoir performance using a systematic application of incorporated, multidisciplinary technologies. It approaches the reservoir as a whole operation, and controls it as a complete system, rather than just a combination of unconnected functions. As such, it is a scheme for applying multiple technologies in an optimum way to achieve synergy.

The uncertainties in any reservoir study are resultants from our incapability to fully describe the reservoir's forces and the flow processes inside it, but we always seek the ultimate minimization of those uncertainties. Doing so, allows us to make the best decisions regarding the strategies and methodologies to be implemented in the reservoir at hand.

Reservoir Simulation is considered one of the most crucial steps in the reservoir management processes. It allows for various options and approaches to be tested and evaluated, without actually spending money in the field or consuming any valuable resources. The main concept of a Reservoir Simulation Software, is to use a descriptive mathematical model of the actual physical parameters that govern the fluid flow in the reservoir. That model should behave in the same manner as the actual reservoir would do (or to the nearest possible approximation).

From these Reservoir Simulation Softwares, we could run as many different scenarios as we want, analyze the performance of the reservoir or forecast its future production under various production mechanisms, we can also implement and execute a variety of solutions. With these data, we can then compare the output of each case to determine which one will be the best to be executed in reality. Hence, the accuracy of these simulation runs and how close they are to the reality is significantly important and irreplaceable.

There are continuous efforts and work to improve the accuracy of the reservoir simulation softwares, and to make them better in predicting the actual oil and gas field performance. This work is done in different ways, from the optimization of the solving methods, to the modification of the original equations that governs the behavior of the fluid flow in the porous media and the mathematical models themselves.

This thesis focuses on the mathematical models development, and how they could be better representing our actual oil field reservoirs. Usually, these models are based on partial differential equations (PDEs) that are solved in different ways. Our work is done on gas flow that follows the non-Gaussian diffusion (anomalous diffusion) pattern, which has been modeled using fractional diffusion equations.

Different solutions (analytical and numerical) have been derived for these fractional diffusion equations, most of them have been produced for the general diffusion equation not specifically for the flow in the porous media. Therefore, the target of this work is to produce an implicit solution to the fractional differential equation describing the anomalous diffusion of single-phase gas flowing in reservoirs.

To accomplish this goal, we need to look at the fundamental laws and equations that governs the flow in porous media, and then modify them to match the target of our work.

1.2 Flow in Porous Media

The flow through porous media is known to be affected by both the media in which it flows, the fluid and all its associated patterns. The laws that governs this flow, combines the parameters and effects of both factors (media and fluid) in order to be able to best describe the flow. This subject (fluid flow through porous media) combines different topics such as fluid dynamics, thermodynamics, applied mathematics, chemistry and geology. In addition, the wide scope it has and its involvement in difficult physical processes make the relevant equations and laws a bit confusing.

There are different laws that describe the flow in porous media; most of them are based on the empirically derived Darcy’s Law (Henry Darcy, 1856).

1.3 Classical Darcy's Law

The classical equation describing the flow of fluid through porous media relating pressure gradient and fluid flux was formulated by Henry Darcy in 1856. This law was developed as a result of experiments on flow of water through sand beds. According to this law, flux is directly proportional to the gradient of the pressure. Darcy's Law is seen as an expression of conservation of momentum, and the following expression could be seen as its main statement. According to the original experiment, it was deduced that: “The rate at which water flows through the filter bed is in direct proportionality to the area of the sand, and also to the difference in height between the inlet and outlet of the bed, and it is inversely proportional to the thickness of that bed”.

Darcy’s law is an empirical relationship which takes the following form for one-dimensional single-phase flow (Ertekin et al, 2001):

$$u = -\frac{k}{\mu}\frac{\partial p}{\partial x} \tag{1.1}$$

For easier interpretation, the same equation could be written as:

$$-\frac{\partial p}{\partial x} = \frac{\mu}{k}u \tag{1.2}$$

Where k is the formation permeability, u is the fluid velocity and μ is the fluid viscosity. The Permeability property combines the geometric properties of the porous media; porosity, grain size, shape of the grains, and cementation degree. It is based on the structure of the porous media and it is completely independent of the fluid nature. The dynamic properties of the porous media is wholly characterized by its permeability value with respect to flow of fluids though it.

The development of the Darcy flow model makes some assumptions that could be summarized as following:

A. A large surface area is exposed to fluid flow in a porous medium, which will lead the viscous resistance to significantly exceed the fluid's acceleration forces, unless turbulence sets in.
B. Laminar or viscous flow is assumed; means the inertia term (fluid density) is not involved. Hence, the fluid's inertia or acceleration forces are being neglected.

1.4 Darcy-Forchheimer Equation (Non-Darcy Flow)

The empirical flow model of Darcy represents a simple linear relationship between the pressure drop and corresponding flow rate in a porous media; and if any deviation from this linearity scenario happens, we term it to be a non-Darcy flow.

It was found that not all fluids flow in the same manner as described by Darcy's law; there are many deviations from this law due to the different assumptions implied in it.

These deviations could be classified as follows:

A. Non-Newtonian fluids
B. High velocity flow
C. Molecular effects

However, the high flow rate effect is the most common phenomenon in petroleum engineering. That forces the fluid to deviate from the assumed laminar flow regime and falls into the turbulent flow area. The following examples are real petroleum reservoirs cases where such effect is seen:

A. Gas reservoirs
B. Near well bore effects (Perforations or Gravel pack system)
C. Fractured reservoirs (Hydraulically or Naturally)

For such cases, it was very important to develop a more practical flow model, which can better represent and characterize the variables and physical parameters encountered.

Philippe Forchheimer, in 1901, discovered that the relationship between potential gradient and flow rate is a non-linear one at sufficiently high velocities, and that this non-linearity is increasing with flow rate. It was found that it is proportional to a new term accounting for an additional pressure drop.

Inertial losses mainly caused that additional pressure drop, and primarily it was due to the acceleration and deceleration effects of the fluid; during its journey through the twisted flow paths of the porous media.

Thus, Forchheimer empirical flow model traditionally stated the total pressure drop to be given by:

$$-\frac{\partial p}{\partial x}=\frac{\mu}{k}u+\beta\rho u^{2} \quad (1.3)$$

where β (beta) is called the inertial factor and the density of the fluid flowing through the medium is given by ρ (rho). The new non-Darcy coefficient “β” appeasers and is given the unit of (inverse length), a lot of authors and researchers attributed its value to be dependent on the porous media characteristics (as a function of permeability and porosity).

Forchheimer’s equation(1.4), can be rewritten in the following form (Forchheimer, 1901):

$$u=-\frac{k}{\mu}\delta\frac{\partial p}{\partial x} \quad (1.4)$$

Where:

$$\delta=\left\{1+\frac{c_{2}k\,\beta\boldsymbol{\rho}\boldsymbol{u}}{|\mu|}\right\}^{-1} \quad (1.5)$$

and c_2 is a conversion factor.

It is clear that this Forchheimer’s equation assumes the validity of Darcy’s equation, but it adds an additional term to account for the increased pressure drop due to the inertial losses at high speeds.

1.5 History Matching

The history matching procedures in the reservoir simulation filed is very critical and of ultimate importance. The concept behind it is to make sure that the model used in representing the reservoir, matches the actual reservoir behavior and can represent it successfully. The task is done by regenerating the old response of the reservoir using the selected mathematical model and compare it against the actual data that is measured from the reservoir. We then keep changing the parameters which are used to generate the response, until an acceptable match is obtained. At this point, we assume that the parameters used to generate the reservoir response are the actual parameters in the reservoir (or the nearest to them). Hence, we can use these parameters to generate a future prediction of the reservoir behavior.

This procedure is tedious and time consuming, as it involves inverse model solutions and calculations. An optimized algorithm should be selected carefully to minimize the time and improve the accuracy of the calculations.

In our case, as we don't have a real field data to compare against, a synthetic dataset is used. We first choose some random parameters to generate a pressure response using our mathematical model. Then we add some noise to that response to mimic a real data gathered from the field. After that, we use our model again and try to find the original parameters used to generate the pressure response (that we assume is the real data), with no input of these original parameters into this history matching process. The outcome of the process will tell us if we were able to obtain a good estimate of the original parameters using the pressure response only.

1.6 Impact of Thesis

The knowledge and results gained from this work will be valuable in building new mathematical models to be used in modeling non-Darcy flow with more realistic equations. The new model will be more representative and better describing the fluid flow in porous media. The findings from this work will be possible reasons to elaborate known inconsistencies in the fluid flow through porous media (especially the turbulent flow regime), and will help to build a new fundamental model for reservoir simulation softwares.

This work is structured into six chapters; each chapter’s content is summarized as follows.

Introduction; the fundamental principles of fluid flow in porous media are covered, with a review of principal equations of flow in porous media.

Literature review; non-Darcy flow in different scenarios is assessed with its basic modeling schemes, with a review of non-Darcy flow modeling in the literature and how to solve them.

Problem Statement; the problem of this work is presented, with the objective and significance of its solution to the petroleum industry.

Solution Statement; the procedure and declaration of developing the proposed solution to the specified problem, and why this approach is the most appropriate one. It also gives some insights of using this procedure.

Analysis and Discussion; a list of results obtained during numerical simulations of gas reservoir and the wells inside them.

Conclusions and Recommendations; the results obtained are discussed and compared with current solutions.

CHAPTER 2
LITERATURE REVIEW

2.1 Flow in Porous Media and Darcy’s Law

As mentioned before, the good understanding of the fluid flow in porous media is very important to better manage the oil reservoirs and the way we produce them. As the gap between energy demand and supply increases, new methods and strategies are required to gain more hydrocarbon out of the already know reserves (in addition to finding new ones).

Although the great effort put in that field, the physics that governs the movement of fluids in a porous media is still questionable and not a straight forward task. In addition, the movement of fluids in a porous media cannot be directly visualized in certain scenarios. Many authors (Biot, 1941, 1956a, 1956b, 1973; Biot and Willis, 1957; McNamee and Gibson, 1960; Bell and Nur, 1978) derived different form of useful equations for diffusion of fluid and their solutions in many interesting cases. However most of the authors mentioned assumed empirically derived Darcy’s Law and formulated their equations of diffusion based on it.

Darcy’s Law (formulated by Henry Darcy in 1856) has many analogies; it is comparable to Ohm's Law for the Conduction of Electricity, Fourier's expression for the conduction of heat or Fick's law in diffusion theory (Hubbert, 1956). This law forms the scientific basis of permeability of the medium that remains constant with time in case of Darcy’s flow.

2.2 Non-Darcy Flow

From many observation, it is clear that some flow behavior does not follow the Darcy's law trend while moving through the porous media. In fact these behaviors contradict with the classic theory of diffusion of pressure and fluids in the porous media. These phenomena might cause the permeability of the system to change such as fluid may carry solid particles that caused pore plugging or chemical reaction with other minerals can change the permeability of the system. It has been experimentally proved that when a fluid flows through a porous medium the permeability of the matrix may be locally variable in time (Caputo, 2000; Iaffaldano et al., 2006; Cloot and Botha, 2006) for the several reasons mentioned above.

It has been also observed that modern diffusion equation fails to describe the behavior of subterranean water in flow through porous media. However most of the research has been done while considering the diffusion of flux rather than the flux of the fluid (Christakos et al., 1995; Mainardi et al., 1996). The main difficulty arises in computing the flux with constant pressure at the boundary because of mathematical computations. So the diffusion of flux requires more attention and a different approach.

In order to describe the flow behavior of fluids, one needs the modification of Darcy's law (or Forchheimer's equation in case of gas flow) by introducing general memory formalisms terms on the flow and pressure gradient as well. Diffusion equation will also require some modifications; so memory formalism was introduced as rheology in the fluid. These memory formalisms are defined as fractional derivatives (Caputo, 2006).

2.3 Anomalous Diffusion

Different sciences use the concept of diffusion: physics, transport phenomena, biological sciences etc. Classical Fick's law can help in modelling the normal diffusion, stating that the diffusion flux is directly proportional to the negative concentration gradient.

In certain cases, some complex objects are produced due to the unexpected movement of particles (Afananasiev et al., 1991) and hence, the probability distribution of these particles cannot be presented by Gaussian distribution during the diffusion processes. Which leads to the difficulty of modeling this movement by classical diffusion equation based on Fick's law.

Several authors described the complex situations that can be described by the use of fractional derivatives (Compte, 1996; Benson et al., 2000; Benson et al., 2001; Del-Castillo-Negrete et al., 2003; Meerschaert, 2002; Metzler and Klafter, 2000).

2.4 M. Caputo Definition

This definition is very popular to be taking the following form for a differentiable function of order n

$$\frac{\partial^{\gamma}}{\partial t^{\gamma}} f(t) \equiv \frac{1}{\Gamma(n-\gamma)} \int_0^t d\tau \frac{1}{(t-\tau)^{1+\gamma-n}} \frac{d^n f(\tau)}{d\tau^n}, \quad n-1 < \gamma < n, (n = \text{integer}) \tag{2.1}$$

2.5 Non Linear Regression

The use of nonlinear regression algorithms in well test analysis for estimating reservoir and well bore parameters was introduced by Rosa and Horne (1983). Following are the some advantages of nonlinear regression over old techniques that make it to use widely today in well test analysis for parameters estimation:

1. Nonlinear regression can interpret uninterpretable tests i.e. it can be applied for any possible reservoir models by generating the corresponding pressure transient solution.

2. Nonlinear regression can analyze multirate or variable rate tests. For these types of variable rate tests, pressure response is calculated for a constant rate production drawdown test based on the reservoir model. After getting the solution, superposition principle is applied to compute the pressure response for an arbitrary flow rate history.

3. The method avoids inconsistent interpretations hence the results are free from human bias.

4. Nonlinear regression provides confidence estimates on answers in conjunction with statistical inference.

2.6 Levenberg-Marquardt Algorithm

In this work, the Levenberg-Marquardt (LM) algorithm is used for nonlinear regression that finds out the minimum of the objective function that is expressed as the sum of squares of non-linear real-valued functions

(Levenberg K. 1944). This technique is considered as a standard for non-linear least-squares problems (Mittelmann, H.D. 2004).

The Hessian matrix H for standard Newton inverse analysis method can be defined as the second derivative of the objective function.

So, the Hessian of the objective function can be written as the following formula states:

$$H(\vec{\alpha}) = S^T C_D^{-1} S + C_M^{-1} + \nabla S^T C_D^{-1} (\vec{d}_{cal} - \vec{d}_{means}) \tag{2.2}$$

where ΔS is the second derivative matrix and it can be given as,

$$\nabla S = \frac{\partial S}{\partial \vec{\alpha}^T} = \frac{\partial^2 \vec{d}_{cal}}{\partial \vec{\alpha} \partial \vec{\alpha}^T} \tag{2.3}$$

In above equation, the Hessian matrix should be positive-definite at each iteration to meet the convergence; because when it is positive-definite the Newton approach yields a downhill direction and meet the quadratic convergence in the neighborhood of the actual solution α. If Hessian matrix is close to singular i.e. not positive-definite then convergence or optimum solution may not be reached.

$$0 < \vec{s}^T H \vec{s}^T < -\vec{g}^T \vec{s} \tag{2.4}$$

In order to solve for s, we need to compute gradient and Hessian at each iteration. The gradient for the objective function defined previously can be represented as,

$$\vec{g}(\vec{\alpha}) = S^T C_D^{-1} (\vec{d}_{cal} - \vec{d}_{meas}) + C_M^{-1} (\vec{\alpha} - \vec{\alpha}_{pri}) \tag{2.5}$$

Where S is the sensitivity matrix which is given as,

$$S = \frac{\partial \vec{d}_{cal}}{\partial \vec{\alpha}} \tag{2.6}$$

The calculation of exact Hessian is computationally expensive and takes very long time. The Levenberg-Marquardt method (Levenberg, 1944; Marquardt, 1963) approximates Hessian matrix to be equal to the diagonal matrix. So the equation becomes,

$$H_{LM}(\vec{\alpha}) = S^T C_D^{-1} S + C_M^{-1} + \dot{\lambda} I \tag{2.7}$$

where λ is a scalar quantity that is multiplied with the identity matrix I of the Hessian which makes it to be always positive definite. This diagonal perturbation will shift every eigenvalue of the Gauss-Newton Hessian by the value of λ. Any eigenvalue that is negative or too close to zero, becomes positive, using this diagonal perturbation. This also improves the condition number of matrix. This perturbation is not limited to Gauss Newton Hessian, but can even be applied to exact Hessian if it is close to singular.

The Levenberg-Marquardt method is a combination of the Gauss–Newton algorithm and the method of steepest descent. When the current solution is far from the correct one, the algorithm behaves like a steepest descent method that slows the convergence rate and sometimes cannot reach to the minimum point. When the current solution is close to the correct solution, it becomes a Gauss-Newton method.

2.7 Memory Effect and Fractional Derivatives

Fractional Diffusion Equations have the focus in many literature work and many authors worked on solutions to solve them, and used the theory in different applications.

Fractional derivatives have been used previously in study of electric transmission lines (Heaviside, 1892), to describe ultrasonic wave propagation physics in human cancellous bone (Sebaa et al., 2006). A new technique for the modeling of speech signal was developed based on fractional integration (Khaled Assaleh and Wajdi Ahmad, 2007).

Fractional derivatives in time can provide improve description of behavior of sound waves in rigid porous materials (Fellah and Depollier, 2002). Also fractional derivatives are useful in modeling of different viscoelastic materials that exhibit complex elastic moduli (Soczkiewicz, 2002).

(Caputo, 1999) used the fractional order differential equations to model the flow of ground water as he modified the law of Darcy by introducing a formalism of a fractional order derivative to represent memory and simulate the effect of permeability reduction with time. A Cloot and JF Botha (2006) also worked on ground water problem putting a complementary derivative (fractional/non-integer order derivatives) in replacement for the classical first order derivative of the piezometric head. This area was also investigated by (Iaffaldano et al, 2006 and Giuseppe et al, 2010).

The anomalous diffusion theory was also used to study the flow in fractured porous media (Chang and Yortsos, 1990; Park et al, 2000, 2001)

2.7.1 Analytical Solutions

(Ji-Huan He, 1998) proposed a new and more precise fractional derivatives model for seepage flow in porous media , which overcome the continuity assumption of seepage and modified the Darcy law. Other authors (Metzler and Klafter, 2000a,b) solved the fractional diffusion equation for diverse boundary value problems, like absorbing and reflecting boundaries in half-space and in a box.

2.7.2 Numerical Solutions

(Meerschaert and Tadjeran, 2004; Yuste and Acedo, 2005; Zhuang and Liu, 2006; Chen et al, 2007; Liu et al, 2007; Murio, 2008) All of them worked on numerical solutions for the fractional diffusion equation. Although not all of them were addressing the flow in porous media, the solutions were of particular interest in our work.

In 1989, Shneider and Wyss obtained Green Functions in closed form for arbitrary space dimensions in terms of Fox functions for time fractional diffusion-wave equations.

Langlands and Henry (2005) presented stability analysis of some numerical methods for time fractional differential equations.

In 2009, an anomalous sub-diffusion equation was discussed by (Liu et al), where there was anomalous decline with time. In addition, Murillo and Yuste (2011) worked on a time derivative in the form of Caputo fractional derivative appears in fractional diffusion and fractional diffusion-wave equations, and they provided an explicit finite difference method for solving them.

Fractional order time derivative and space derivative are somewhat different in describing the physics of the flow. This concept is well defined and presented by Caputo (Caputo, 2002). If modeling of local perturbation is concerned then fractional order time derivative will be useful, however if variations in an infinite medium is to be captured then fractional order space derivatives are appropriate i.e. flow will be related to memory by recalling the path of pressure gradient from the beginning of the flow.

To include the effect of wellbore storage and skin in presence of memory, new mathematical procedures and a generalized form of bottom hole pressure was formulated (Park, et al., 2001). In this paper, a new solution is derived and analyzed for bottom-hole pressure distribution which permits the wellbore storage and skin effects for fractal reservoirs. After that, a general mathematical formula is proposed for the analysis of pressure behavior in the case of three-dimensional anisotropic reservoir. Also Ali and Malik (2014) worked on Hilfer fractional advection–diffusion equations with power-law initial condition; as a numerical study using variational iteration method.

CHAPTER 3

PROBLEM STATEMENT

3.1 Current Limitations

The diffusivity equations which governs the flow in the porous media is based/derived from the three fundamental laws which are:

(a) Law of conservation of mass or the continuity equation

(b) Equation of state of the fluid

(c) Law governing the dynamics of fluid flow or Newton's law

These laws, and hence the diffusivity equation, do not account for the memory effect of the flow, which means that the anomalous flow is not properly modeled using it. Thus, the diffusivity equation needs to be modified to account for the new memory effect of the flow during its journey in the porous media.

This modification is proposed to be the fractional differential equations to add a new parameter (alpha α) to control the presence of the memory formalism.

This addition is usually easy and straight forward in case we are dealing with the classical Darcy's law, it is easy to be implemented in those cases of Gaussian flow that obeys Darcy's law by just changing the $\frac{\partial p}{\partial x}$ term with $\frac{\partial}{\partial x}(\frac{\partial^{\alpha} p}{\partial t^{\alpha}})$ (Caputo, 1999). But in case of Forchheimer's equation, additional work is required. We are dealing with gas, using Forchheimer's Equation.

We will need to add more work as the relationship here is not linear between the gradient and the flow.

3.2 Work Objective

The main objective of this work is to implement the memory effect formalism into the flow equation of turbulent gas flow. That has been achieved using fractional diffusion equations, to be able to describe the anomalous diffusion in porous media for gas.

This model is tested by solving a history matching case (with and without the memory effect) and then comparing the outcomes. In addition, a modified flow model handling the flow between the gridblock and the wellbore is presented to help calculating the wellbore pressure from the gridblock pressure.

CHAPTER 4
MATHEMATICAL MODEL

4.1 Proposed Solution Methodology

As shown in the literature review section, different formalisms are proposed to represent the memory formalism and its effect. In this work, to solve the anomalous flow problem, the Caputo's definition of fractional derivative will be used combined with the L1 formula (Oldham, K. B. and Spanier, J, 1974) to discretize the fractional derivative. Reservoir model will be built, programmed and verified with different values of the fractional derivative α.

Fractional Derivative in Caputo's sense (Caputo, 1999):

$$\frac{\partial^\gamma}{\partial t^\gamma} f(t) \equiv \frac{1}{\Gamma(n-\gamma)} \int_0^t d\tau \frac{1}{(t-\tau)^{1+\gamma-n}} \frac{d^n f(\tau)}{d\tau^n}, \quad n-1<\gamma<n, (n = \text{integer}) \tag{4.1}$$

where γ is the fractional order of the F function

We use the L1 Formula for purpose of discretization (Oldham, K. B. and Spanier, J., 1974) as following:

$$\left.\frac{\partial^\gamma f}{\partial t^\gamma}\right|_{t_m} \simeq \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)} \sum_{k=0}^{m-1} b_k^{(1-\gamma)} \left[f(t_{m-k}) - f(t_{m-1-k}) \right] \tag{4.2}$$

Where m represent each gridblock for the discretization.

the value of *b* is given by

$$b_k^{(1-\gamma)} = \left(k\ +1\right)^{1-\gamma} - k^{\,1-\gamma} \tag{4.3}$$

4.2 Mathematical Model Derivation

As stated previously, the diffusivity equation is derived from the three principle laws. To derive the modified diffusivity equation (that accounts for the memory effect for gas), we will start from the mass conservation law (for one dimension):

$$(\dot{m}_{in} - \dot{m}_{out}) + \dot{m}_s = \dot{m}_{acc} \tag{4.4}$$

$$\left[\left(\rho uA\right)_{in} - \left(\rho uA\right)_{out}\right] + \frac{q_m}{\alpha_c} = \frac{1}{\alpha_c}\frac{(v_b\phi\rho)_{t+\Delta t} - (v_b\phi\rho)_t}{\Delta t} \tag{4.5}$$

$$\left[\left(\rho u_x A_x\right)_{h-1/2} - \left(\rho u_x A_x\right)_{h+1/2}\right] + \frac{q_m}{\alpha_c} = \frac{1}{\alpha_c}\frac{(v_b\phi\rho)_{t+\Delta t} - (v_b\phi\rho)_t}{\Delta t} \tag{4.6}$$

$$\frac{-\partial}{\partial x}\left[\rho u_x A_x\right]\Delta x + \frac{q_m}{\alpha_c} = \frac{v_b}{\alpha_c}\frac{\partial}{\partial t}\left(\rho\phi\right) \tag{4.7}$$

The equation (4.7) has no special annotation, we write it in terms of gas properties as follows, and it becomes

$$\frac{-\partial}{\partial x}\left(\rho_g u_x A_x\right) + \frac{q_m}{\alpha_c} = \frac{v_b}{\alpha_c}\frac{\partial}{\partial t}\left(\rho_g\phi\right) \tag{4.8}$$

If we look at equation(4.8), we will see that all the parameters could easily be supplied, except for the velocity term (u_x). To determine the value of the velocity (u_x), we refer to the Original Darcy Law as follows:

$$u_x = \frac{-B_c k}{\mu} . \frac{\partial p}{\partial x} \tag{4.9}$$

$$\frac{-\partial p}{\partial x} = \frac{\mu}{B_c k} u_x \tag{4.10}$$

But since we deal with gas, we use the modified Darcy law known as Forchheimer equation (by adding the non-Darcy term β). A key to applying the Forchheimer equation is to estimate a value for β. Methods developed to calculate β are based on experimental work, correlations, and from the Forchheimer equation itself that can be stated as follows:

$$\frac{-\partial p}{\partial x} = \frac{\mu}{k} u_x + \beta \rho u_x^2 \tag{4.11}$$

Units of the parameters are: p {atm}, k {Darcy}, L {cm}, ρ {gm/cc}, μ {cp} u {cm/sec}, β {atm-sec2/gm}

where β is the non-Darcy flow coefficient. Dimensional analysis reveals that β has dimensions of 1/length; therefore the conversion factor is given by,

$$\beta \left[\frac{1}{ft} \right] = \beta \left[\frac{atm - \sec^2}{gm} \right] * 3.0889 x 10^7 \tag{4.12}$$

For the equation to be in the oil field units, we recall that:

$$\frac{-\partial p}{\partial x} = \frac{\mu}{k} \frac{q}{A_x} + \beta \rho \left(\frac{q}{A_x} \right)_x^2$$
$$\beta = \left[\frac{atm - \sec^2}{gm} \right] \tag{4.13}$$

The same equation could be written in the form of some coefficients and then takes the following form:

$$\frac{-\partial p}{\partial x} = \frac{1}{C_1}\frac{\mu}{k}\frac{q}{A_x} + \frac{C_2}{C_1}\beta\rho\left(\frac{q}{A_x}\right)_x^2 \tag{4.14}$$

Where

$$C_1 = \frac{1}{14.7}\frac{1}{30.48}\frac{1}{1000}30.48^2\frac{24*60*60}{5.615*30.48^3} = 1.127E-03 \tag{4.15}$$

Hence $C_1 = B_c = 1.127\text{x}10^{-3}$

$$C_2 = \frac{1}{62.4}30.48^2\frac{5.615*30.48^3}{24*60*60}\frac{1}{1000}\frac{1}{30.48^4} = 3.17443E-08 \tag{4.16}$$

For beta to be in 1/ft unit, C2 should be divided by $3.0889\text{x}10^7$ as shown previously from dimensional analysis.

We may calculate the final conversion factor for beta that will be

C_3 $= C_2 / (3.0889\text{x}10^7 * C_1) = 3.17443\text{x}10^{-8} / (3.0889\text{x}10^7 * 1.127\text{x}10^{-3})$

$= 9.12\text{x}10^{-13}$ (4.17)

And finally put the equation as:

$$\frac{-\partial p}{\partial x} = \frac{\mu}{B_c k}u_x + C_3\beta\rho u_x^2 \tag{4.18}$$

The units in this form will be as follows:

$$\left[\frac{psi}{ft}\right]=\left[\frac{\sec}{ft^{2}}\right]\left[\frac{1}{mDarcy}\right]\left[\frac{ft}{\sec}\right]+\left[\frac{1}{ft}\right]\left[\frac{lb}{ft^{3}}\right]\left[\frac{ft^{2}}{\sec^{2}}\right] \tag{4.19}$$

To put the equation if the terms of velocity, we can rearrange the terms to get a direct expression for the velocity. By doing so, it will be easier for us to substitute in the original equation of mass conservation to finally get the full diffusivity equation for gas.

By taking the velocity as a common factor we get

$$\frac{-\partial p}{\partial x}=\frac{\mu}{B_c k}u_x\left(1+\frac{C_3 B_c k\,\beta\rho|u_x|}{\mu}\right) \tag{4.20}$$

And for the sake of simplicity, we may set:

$$C_4=C_3\times B_c \text{ and } \delta_x=\left(\frac{1}{1+\frac{C_4 k\,\beta\rho|u_x|}{\mu}}\right) \tag{4.21}$$

Hence, we get the final equation as:

$$u_x=\frac{-B_c k}{\mu}\delta_x\frac{\partial p}{\partial x} \tag{4.22}$$

Noting that the term δ_x is dimensionless.

From the General Gas Law, we may replace the density term with its equivalent as follows:

$$\rho_g = \frac{\rho_{s.c.}}{B_g} = \frac{PM}{zRT} \tag{4.23}$$

Then Substituting in the main equation using both velocity and density equations, we get:

$$\frac{-\partial}{\partial x}\left(\frac{\rho_{s.c}}{B_g}.\frac{-B_c k}{\mu_g}\delta_x \frac{\partial p}{\partial x}.A_x\right)\Delta x + \frac{\alpha_c q_{s.c}\rho_{s.c}}{\alpha_c} = \frac{\rho_{s.c}v_b}{\alpha_c}\frac{\partial}{\partial t}\left(\frac{\phi}{B_g}\right) \tag{4.24}$$

$$\frac{\partial}{\partial x}\left(\frac{B_c k_x A_x}{B_g \mu_g}\delta_x \frac{\partial p}{\partial x}\right)\Delta x + q_{s.c} = \frac{v_b}{\alpha_c}\frac{\partial}{\partial t}\left(\frac{\phi}{B_g}\right) \tag{4.25}$$

We try to put all the parameters in the form of pressured-based functions, and for that we need to rewrite the term of the Gas Formation Volume Factor (B_g).

We already have the B_g formula as

$$B_g = \frac{p_{s.c}TZ}{\alpha_c T_{s.c} p} \tag{4.26}$$

By substituting in our last equation and continue reducing, we get the final form of the diffusivity equation for gas flow (without modification)

$$\frac{\partial}{\partial x}\left(\frac{\alpha_c T_{s.c}}{p_{s.c}T}\frac{p}{Z}\frac{B_c k_x A_x}{\mu_g}\delta_x \frac{\partial p}{\partial x}\right) + q_{s.c} = \frac{v_b}{\alpha_c}\frac{\partial}{\partial t}\left(\phi\frac{\alpha_c T_{s.c}}{p_{s.c}T}\frac{p}{Z}\right) \tag{4.27}$$

$$\frac{\alpha_c T_{s.c}}{p_{s.c}T}\frac{\partial}{\partial x}\left(\frac{B_c k_x A_x}{\mu_g}\frac{p}{Z}\delta_x \frac{\partial p}{\partial x}\right) + q_{s.c} = \frac{v_b \phi T_{s.c}}{p_{s.c}T}\frac{\partial}{\partial t}\left(\frac{p}{Z}\right) \tag{4.28}$$

$$\frac{\partial}{\partial x}\left(\frac{B_c k_x A_x}{\mu_g}\frac{p}{Z}\delta_x\frac{\partial p}{\partial x}\right)+q_{s.c}\frac{p_{s.c}T}{\alpha_c T_{s.c}}=\frac{v_b\phi}{\alpha_c}\frac{\partial}{\partial t}\left(\frac{p}{Z}\right) \tag{4.29}$$

$$\frac{\partial}{\partial x}\left(\frac{B_c k_x A_x}{\mu_g}\frac{p}{Z}\delta_x\frac{\partial p}{\partial x}\right)+0.0283\frac{T}{\alpha_c}q_{g,s.c}=\frac{v_b\phi}{\alpha_c}\frac{\partial}{\partial t}\left(\frac{p}{Z}\right) \tag{4.30}$$

The final form in [STB/d] is:

$$\frac{\partial}{\partial x}\left(\frac{B_c k_x A_x}{\mu_g}\frac{p}{Z}\delta_x\frac{\partial p}{\partial x}\right)\Delta x+0.0283\frac{T}{\alpha_c}q_{g,s.c}=\frac{v_b\phi}{\alpha_c}\frac{\partial}{\partial t}\left(\frac{p}{Z}\right) \tag{4.31}$$

4.2.1 Finite Difference Approximation

Up to Equation(4.31), the equation is for gas flow in porous media (and it still does not consider the memory effect). To account for the memory effect, we need to modify the space derivative into a fraction derivative. To accomplish that, we should discretize that equation that we have already deduced into an Implicit and Heterogeneous system.

Let's consider a reservoir that is discretized into M gridblocks, each grid of dimension Δx, Δy and Δz. All the edges of the reservoir is closed (no flow enters or exits through them). Wells acting as sink or source may be drilled in any of the gridblocks.

$$\frac{1}{\Delta x}\left[\left(\frac{B_c k_x A_x}{\mu_g}\frac{p}{Z}\delta_x\frac{\partial p}{\partial x}\right)_{m,m+1}-\left(\frac{B_c k_x A_x}{\mu_g}\frac{p}{Z}\delta_x\frac{\partial p}{\partial x}\right)_{m,m-1}\right]\Delta x \tag{4.32}$$
$$+0.0283\frac{T}{\alpha_c}q_{g,s.c}=\frac{v_b\phi}{\alpha_c}\frac{\partial}{\partial t}\left(\frac{p}{Z}\right)$$

$$\left(\frac{B_c k_x A_x}{\mu_g}\frac{p}{Z}\delta_x\frac{\partial p}{\partial x}\right)_{m,m+1}-\left(\frac{B_c k_x A_x}{\mu_g}\frac{p}{Z}\delta_x\frac{\partial p}{\partial x}\right)_{m,m-1}$$
$$+0.0283\frac{T}{\alpha_c}q_{g,s.c}=\frac{v_b\phi c_g}{\alpha_c}\frac{p}{Z}\frac{\partial p}{\partial t} \tag{4.33}$$

$$\left(\frac{B_c k_x A_x}{\Delta x\,\mu_g}\frac{p}{Z}\delta_x\right)_{m,m+1}^{n+1}(p_{m+1}^{n+1}-p_m^{n+1})-\left(\frac{B_c k_x A_x}{\Delta x\,\mu_g}\frac{p}{Z}\delta_x\right)_{m,m-1}^{n+1}(p_m^{n+1}-p_{m-1}^{n+1})$$
$$+0.0283\frac{T}{\alpha_c}q_{g,s.c}=\frac{(v_b\phi)_m}{\Delta t\,\alpha_c}\left(\frac{c_g p}{Z}\right)_m^{n+1}(p_m^{n+1}-p_m^{n}) \tag{4.34}$$

4.2.2 Memory Formalism Implementation

In this section, we show how to deal with the flow in case the rate of diffusion was anomalous, we introduce a memory formalism parameter to account for this flow.

This change will be shown on both the original equation affecting the units of both the permeability and the non-Darcy flow coefficient (beta). Also the change will carry on while we complete the discretization of the diffusivity equation for the numerical solution.

In the previous section, we have already stated that

$$u_x=\frac{-B_c k}{\mu}\delta_x\frac{\partial p}{\partial x} \tag{4.35}$$

This equation holds in case of normal diffusion, but as we are dealing anomalous diffusion, new parameters will be introduced to account for this effect. A modified Darcy’s law that describes the flow of fluid in a reservoir with long memory us given by (Caputo, 1999):

$$u = \frac{-\hat{k}}{\mu}\frac{\partial}{\partial x}\left(\frac{\partial^{\alpha} p}{\partial t^{\alpha}}\right) \tag{4.36}$$

Where α is the fractional order of differentiation in the sense of Caputo. A pseudo-permeability k, with unit md/s^{α}, is used instead of the original permeability term k, with the units of md, which governs the consistency of the equation's units.

Equation(4.36) is valid for the normal Darcy's law, but since we are dealing with gas and implementing the Forchheimer's equation, we need to further modify our equation to be in the form of

$$u = \frac{-B_c\hat{k}}{\mu}\hat{\delta}_x \frac{\partial}{\partial x}\left(\frac{\partial^{\alpha} p}{\partial t^{\alpha}}\right) \tag{4.37}$$

And finally written as

$$u = \frac{-B_c\hat{k}}{\mu}\delta_x \frac{\partial \hat{p}}{\partial x} \tag{4.38}$$

where

$$\hat{p} = \frac{\partial^{\alpha} p}{\partial t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{\partial p(x,\tau)}{\partial \tau}, 0 \le \alpha < 1 \tag{4.39}$$

$\hat{k}$ has the units of md/s^{α}

And for δ to remain dimensionless, the units of β will be changed into $\hat{\beta} = \frac{s^{\alpha}}{ft}$ as δ has the multiplication of $k*\beta$ which now becomes $\hat{k}*\hat{\beta}$

$$\delta_x = \left(\frac{1}{1 + \frac{C_4 \hat{k} \hat{\beta} \rho |u_x|}{\mu}} \right) \quad (4.40)$$

$$\hat{\beta} = 1.88 * 10^{10} \hat{k}^{-1.47} \phi^{-0.53} \quad (4.41)$$

The boundary conditions of the reservoir in all directions are given by:

$$\frac{\partial}{\partial x} \left(\frac{\partial^\alpha p}{\partial t^\alpha} \right) = 0 \quad (4.42)$$

For the fractional derivative of the pressure, it can be discretized using the L1 formula as

$$\hat{p}_m^{n+1} = \frac{\partial^\alpha p_m^{n+1}}{\partial t^\alpha} = \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \left(p_m^{n+1} - p_m^n \right) + \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{\kappa=1}^{n} b_\kappa^\alpha \left(p_m^{n+1-\kappa} - p_m^{n-\kappa} \right) + O(\Delta t) \quad (4.43)$$

$$with : 0 < \alpha < 1$$
$$b_\kappa^\alpha = (\kappa + 1)^{1-\alpha} - \kappa^{1-\alpha} \quad (4.44)$$

Which could be shorthanded in the form of:

$$\hat{p}_m^{n+1} = D p_m^{n+1} + E_m^n \quad (4.45)$$

where :

$$D = \frac{\left(\Delta t\right)^{-\alpha}}{\Gamma\left(2-\alpha\right)} \tag{4.46}$$

$$E_m^n = \frac{\left(\Delta t\right)^{-\alpha}}{\Gamma\left(2-\alpha\right)}\left[-p_m^n + \sum_{\kappa=1}^{n} b_\kappa^\alpha \left(p_m^{n+1-\kappa} - p_m^{n-\kappa}\right)\right] \tag{4.47}$$

In the discretization process, and for the memory effect to be implemented, we replace all the pressure values that change with space (left hand pressure values), to be changing with both time and space. Hence, we will use dotted pressure ($\dot{p}$) for that notation, and the equation becomes:

$$\begin{aligned}&\left(\frac{B_c \hat{k}_x A_x}{\Delta x\, \mu_g}\frac{p}{Z}\delta_x\right)_{m,m+1}^{n+1}(\hat{p}_{m+1}^{n+1} - \hat{p}_m^{n+1}) - \left(\frac{B_c \hat{k}_x A_x}{\Delta x\, \mu_g}\frac{p}{Z}\delta_x\right)_{m,m-1}^{n+1}(\hat{p}_m^{n+1} - \hat{p}_{m-1}^{n+1})\\ &+0.0283\frac{T}{\alpha_c}q_{g,s.c} = \frac{\left(v_b\phi\right)_m}{\Delta t\, \alpha_c}\left(\frac{c_g p}{Z}\right)_m^{n+1}(p_m^{n+1} - p_m^n)\end{aligned} \tag{4.48}$$

Also if we take the fluid transmissibility as

$$T_m^{n+1} = \left(\frac{B_c k_x A_x}{\Delta x\, \mu_g}\frac{p}{Z}\delta_x\right)_m^{n+1} \tag{4.49}$$

$$T_{m,m+1}^{n+1} = T_{m+1/2}^{n+1} = \frac{2T_m^{n+1}T_{m+1}^{n+1}}{T_m^{n+1} + T_{m+1}^{n+1}} \tag{4.50}$$

The same goes with

$$C_m^{n+1} = \left(\frac{v_b\phi}{\Delta t\, \alpha_c}\frac{c_t p}{Z}\right)_m^{n+1} \tag{4.51}$$

where

$$c_t = c_g S_g + c_w S_w + c_f \tag{4.52}$$

$$c_f = \left(\frac{1.782}{\phi^{0.438}}\right)10^{-6} \tag{4.53}$$

$$\begin{aligned}\mu_g = &-5.3765*10^{-14} p^3 + 6.6632*10^{-10} p^2 + 7.3281*10^{-7} p \\ &+1.3617*10^{-2}\end{aligned} \tag{4.54}$$

$$\mu_g c_g = \frac{3.41985*10^{-2}}{p^2} + \frac{1.30395*10^{-2}}{p} + 2.4189*10^{-6} \tag{4.55}$$

$$c_w = 1/\left(7.033p + 0.5415S - 537.0T + 403{,}300\right) \tag{4.56}$$

Where P is the pressure (psi), T is the temperature (F^o), S is Water Salinity (mg/L). The symbols ct, cg, cc, cf are the total, gas, water and formation compressibility, respectively. Finally, the Sg, Sw are the gas and water saturation, respectively.

Substituting in (4.48) we get:

$$\begin{aligned}&T^{n+1}_{m,m+1}(\dot{p}^{n+1}_{m+1} - \dot{p}^{n+1}_{m}) - T^{n+1}_{m,m-1}(\dot{p}^{n+1}_{m} - \dot{p}^{n+1}_{m-1}) + 0.0283\frac{T}{\alpha_c}q_{g,s.c} \\ &= C^{n+1}_{m}(p^{n+1}_{m} - p^{n}_{m})\end{aligned} \tag{4.57}$$

$$\begin{aligned}&T^{n+1}_{m,m+1}\left(Dp^{n+1}_{m+1} + E^{n}_{m+1} - \left(Dp^{n+1}_{m} + E^{n}_{m}\right)\right) \\ &-T^{n+1}_{m,m-1}\left(Dp^{n+1}_{m} + E^{n}_{m} - \left(Dp^{n+1}_{m-1} + E^{n}_{m-1}\right)\right) \\ &+0.0283\frac{T}{\alpha_c}q_{g,s.c} = C^{n+1}_{m}(p^{n+1}_{m} - p^{n}_{m})\end{aligned} \tag{4.58}$$

The next step will be removing the $\dot{p}$ terms by substituting from the equivalent equation showed previously in terms of E and Dp

$$T_{m,m+1}^{n+1}Dp_{m+1}^{n+1} + T_{m,m+1}^{n+1}E_{m+1}^{n} - T_{m,m+1}^{n+1}Dp_{m}^{n+1} - T_{m,m+1}^{n+1}E_{m}^{n} - T_{m,m-1}^{n+1}Dp_{m}^{n+1} - T_{m,m-1}^{n+1}E_{m}^{n} + T_{m,m-1}^{n+1}Dp_{m-1}^{n+1}\cdots$$
$$+T_{m,m-1}^{n+1}E_{m-1}^{n} + 0.0283\frac{T}{\alpha_c}q_{g,s.c} = C_{m}^{n+1}p_{m}^{n+1} - C_{m}^{n+1}p_{m}^{n}$$

We continue rearranging the equation to take the final form in which the equations will be solved numerically

$$\begin{aligned}
&T_{m,m-1}^{n+1}Dp_{m-1}^{n+1} - T_{m,m-1}^{n+1}Dp_{m}^{n+1} - T_{m,m+1}^{n+1}Dp_{m}^{n+1} - C_{m}^{n+1}p_{m}^{n+1} + T_{m,m+1}^{n+1}Dp_{m+1}^{n+1} \\
&= -C_{m}^{n+1}p_{m}^{n} - 0.0283\frac{T}{\alpha_c}q_{g,s.c} \\
&\quad -T_{m,m-1}^{n+1}E_{m-1}^{n} + T_{m,m-1}^{n+1}E_{m}^{n} + T_{m,m+1}^{n+1}E_{m}^{n} - T_{m,m+1}^{n+1}E_{m+1}^{n}
\end{aligned} \tag{4.59}$$

$$\begin{aligned}
&T_{m,m-1}^{n+1}Dp_{m-1}^{n+1} - T_{m,m-1}^{n+1}Dp_{m}^{n+1} - T_{m,m+1}^{n+1}Dp_{m}^{n+1} - C_{m}^{n+1}p_{m}^{n+1} + T_{m,m+1}^{n+1}Dp_{m+1}^{n+1} \\
&= -C_{m}^{n+1}p_{m}^{n} - 0.0283\frac{T}{\alpha_c}q_{g,s.c} \\
&-T_{m,m-1}^{n+1}E_{m-1}^{n} + T_{m,m-1}^{n+1}E_{m}^{n} + T_{m,m+1}^{n+1}E_{m}^{n} - T_{m,m+1}^{n+1}E_{m+1}^{n}
\end{aligned} \tag{4.60}$$

$$\begin{aligned}
&DT_{m,m-1}^{n+1}p_{m-1}^{n+1} - \left[DT_{m,m-1}^{n+1} + DT_{m,m+1}^{n+1} + C_{m}^{n+1}\right]p_{m}^{n+1} + DT_{m,m+1}^{n+1}p_{m+1}^{n+1} \\
&= -C_{m}^{n+1}p_{m}^{n} - 0.0283\frac{T}{\alpha_c}q_{g,s.c} \\
&-T_{m,m-1}^{n+1}E_{m-1}^{n} + \left[T_{m,m-1}^{n+1} + T_{m,m+1}^{n+1}\right]E_{m}^{n} - T_{m,m+1}^{n+1}E_{m+1}^{n}
\end{aligned} \tag{4.61}$$

The equation (4.61) is the final modified diffusivity equation for gas, which considers the memory effect and is designed for the heterogeneous and implicit solution. It has been programmed and used for modelling of gas flow and the results are showing in the next section.

4.2.3 Bottom Hole Pressure Calculation

As shown in (Awotunde, Ghanam, Al-Homidan, and Tatar 2015). We can get the P_{wf} value after solving eq. (4.61) for pressure. We can then calculate the P_{wf} from the gridblock pressure by re-substituting back into forchheimer's equation. Assuming a cylindrical wellbore model in a gridblock pressure of P_{blk}

$$q_{s.c.} = -\frac{2\pi B_c \hat{k} h}{\mu B_g} \delta \frac{\hat{p}_{blk} - \hat{p}_{wf}}{\ln\left(\frac{r_{eq}}{r_w} + skin\right)} \tag{4.62}$$

$$q_{s.c.} = -WI \left(\hat{p}_{blk} - \hat{p}_{wf} \right) \tag{4.63}$$

$$\hat{p}_{wf} = \hat{p}_{blk} + \frac{q_{s.c.}}{WI} \tag{4.64}$$

The previous equation (4.64) assumes the well to be in the center of the gridblock.

The value of the equivalent well-block (gridblock containing the well) radius is the radius at which the steady-state pressure in the reservoir is equivalent to the well-block pressure, and is given by:

$$r_{eq} = \frac{0.28\left(\sqrt{\frac{k_y}{k_x}}\Delta x^2 + \sqrt{\frac{k_x}{k_y}}\Delta y^2\right)^{0.5}}{\left(\frac{k_y}{k_x}\right)^{0.25} + \left(\frac{k_x}{k_y}\right)^{0.25}} \tag{4.65}$$

The previous equation(4.65) would be edited to account for the changing permeability value we have in our model, to be presented as following:

$$r_{eq} = \frac{0.28\left(\sqrt{\frac{k_y}{k_x}}\Delta x^2 + \sqrt{\frac{k_x}{k_y}}\Delta y^2\right)^{0.5}}{\left(\frac{k_y}{k_x}\right)^{0.25} + \left(\frac{k_x}{k_y}\right)^{0.25}} \tag{4.66}$$

CHAPTER 5

ANALYSIS AND DISCUSSION OF RESULTS

5.1 Numerical Computation

From the previously derived equations, the forward model is formulated and run to estimate the pressure distribution in the reservoir over a period of 10 days and with 3 wells distributed in the system.

Realistic data was used to generate the reservoir model, and different equations were used to calculate and interpolate the different gas properties. The data in the Table 1 was used to formulate the reservoir:

Table 1 Parameters used for reservoir formulation

Parameter	**Value**
No. Grids in X direction	16
No. Grids in Y direction	16
No. Grids in Z direction	1
Length in X direction	6400 ft
Length in Y direction	4800 ft
Length in Z direction	60 ft
α	0.2
ϕ	0.15
Temperature	200^{o} F
S_g	0.65
P_i	6000 psi
P_g ref	6000 psi
B_g ref	187.62

For the three wells used in the reservoir, Table 2 shows the properties they had as following:

Table 2 Wells data for reservoir formulation

Well #	X - Location	X - Location	X - Location	Rate (MMscf/day)
1	8	5	1	-4
2	3	3	1	-6
3	15	13	1	-8

The negative sign in the wells rate indicates that they were all producing wells with the specified rates and locations.

The permeability distribution used for this run was ranging from 0.2 to 1.8 md as showing in Figure 5-1

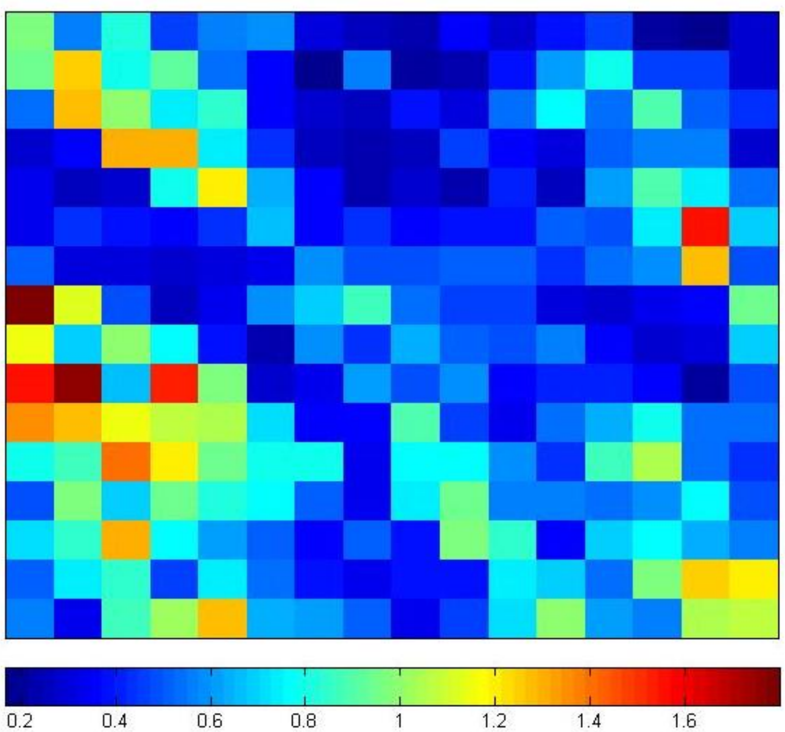

Figure 5-1 16x16 grid perm distribution

To show the effect of the non-Darcy term on the pressure calculation, the graphs (Figure 5-2, Figure 5-3 and Figure 5-4) show a comparison of the bottom hole pressure values with and without the Beta value being calculated, and it is clear from the graphs that the pressure drop caused by the non-Darcy flow term is significant and cannot be ignored. T

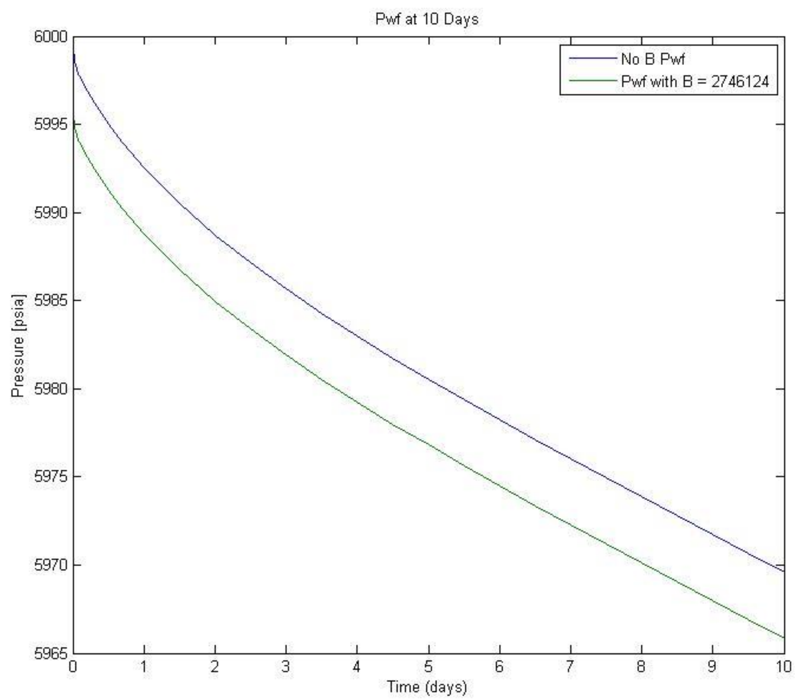

Figure 5-2 Pwf of well no.1

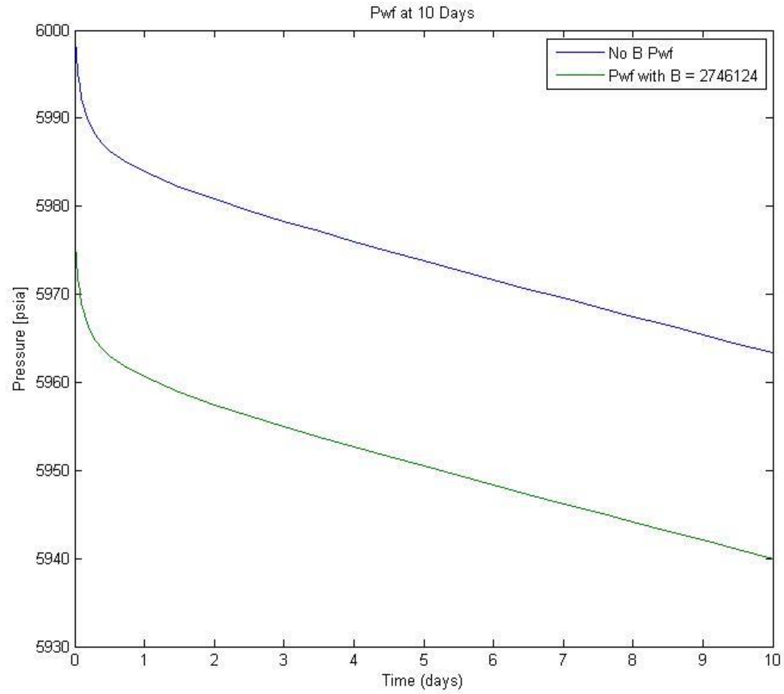

Figure 5-3 Pwf of well no.2

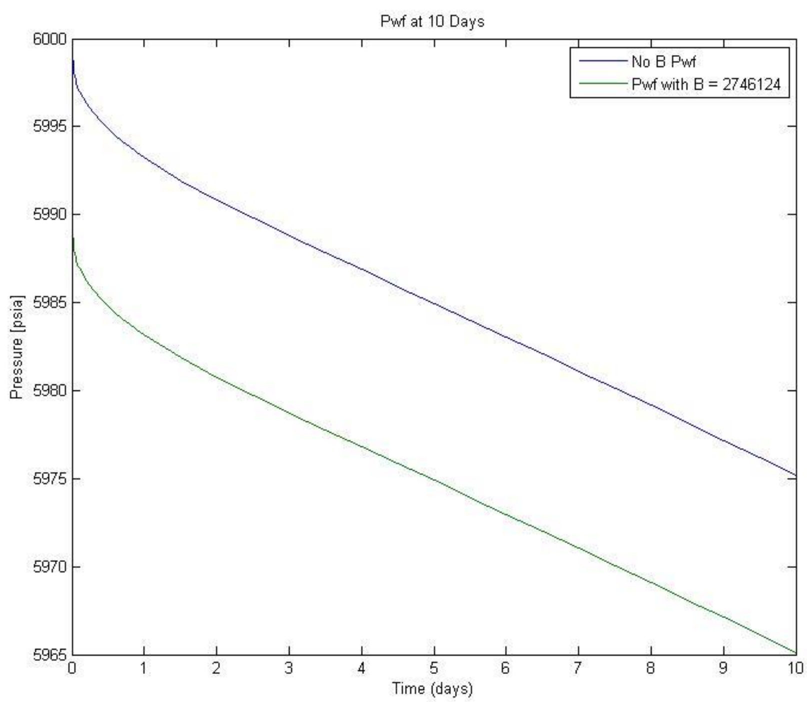

Figure 5-4 Pwf of well no.3

The pressure distribution from the forward model is presented in both 2D (Figure 5-5) and 3D () figures:

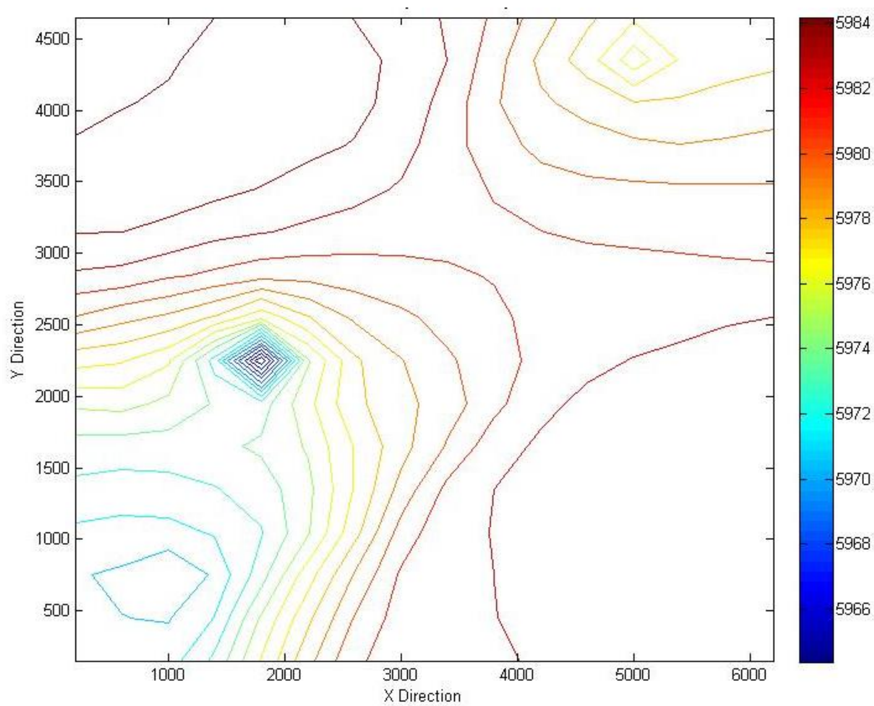

Figure 5-5 Pressure Distribution at 10 Days (top-view)

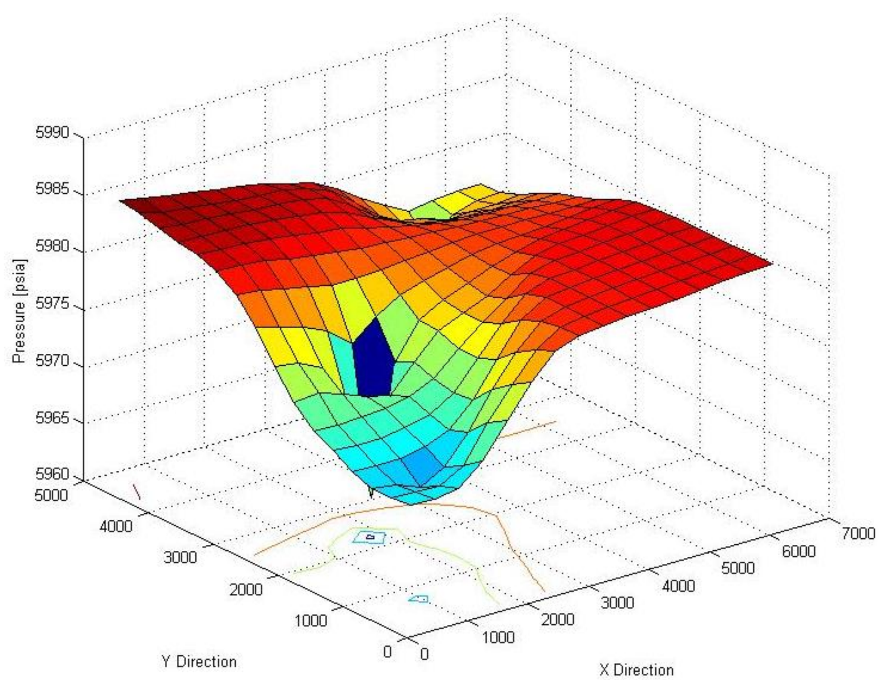

Figure 5-6 Pressure Distribution at 10 days (3D View)

5.2 Sensitivity Studies

5.2.1 Grid System sensitivity

In this section we run a sensitivity study on the size of the system (number of grids), to make sure that the model is valid with different grid sizes and is not directly dependent on the number of grids.

Table 3 shows the different grid systems used and the differences noticed on the pressure drop with the same reservoir size.

Table 3 Sensitivity study on different reservoir dimensions

Grid Size X * Y * Z	Reservoir Dimensions X * Y * Z	Pressure Drop Diff. (10 days same well)
16x16x1	6400x4800x70 ft	+1 psi
24x24x1	6400x4800x70 ft	-2 psi
32x32x1	6400x4800x70 ft	+2 psi
40x40x1	6400x4800x70 ft	+1 psi
48x48x1	6400x4800x70 ft	-1 psi

And they seem to be rounding errors and that the model is independent of the grid size used.

5.2.2 Alpha Value sensitivity

It is necessary to check the effect of the exponent of the fractional derivative (α) on the system and the pressure drop in the wells.

In the following graphs (from Figure Figure 5-7 to Figure 5-12), show the change in the exponent of the fractional derivative (α) between two extremes (from 0.00 to 0.10) to see the effect it has on the pressure drop in different wells.

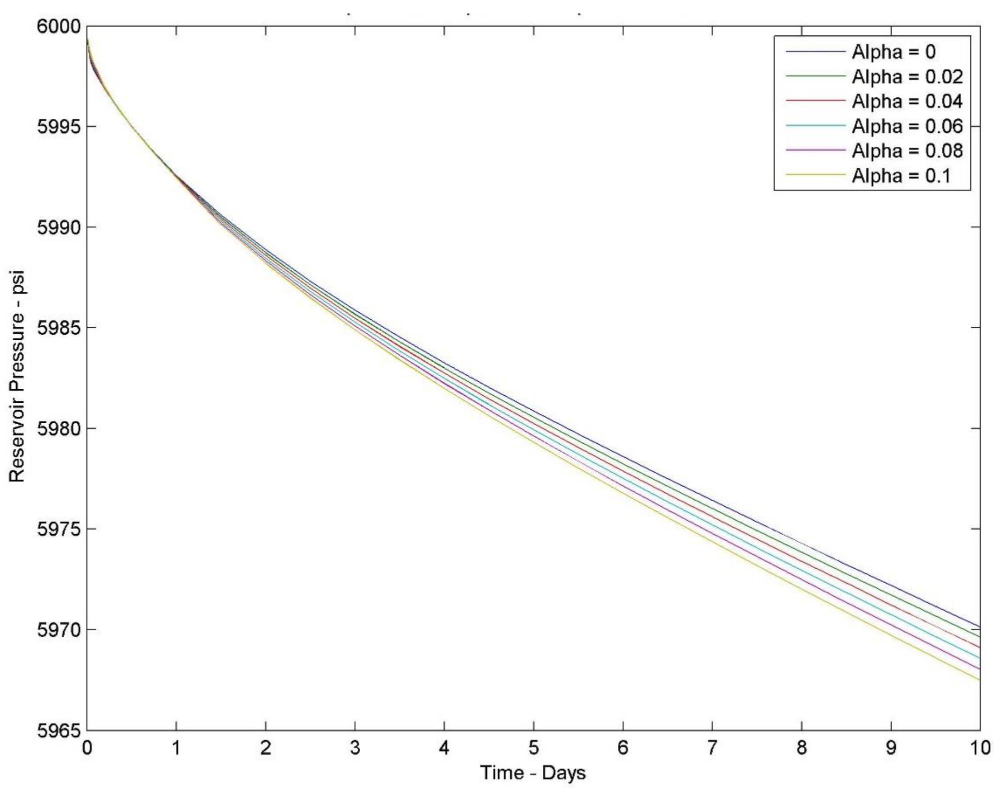

Figure 5-7 Grid Pressure of well no.1 change with alpha

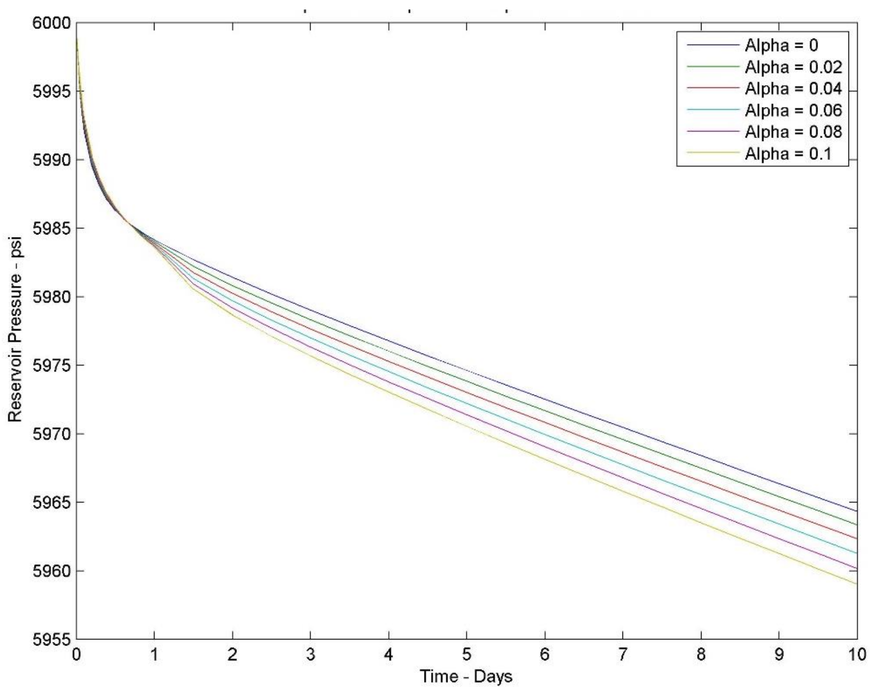

Figure 5-8 Grid Pressure of well no.2 change with alpha

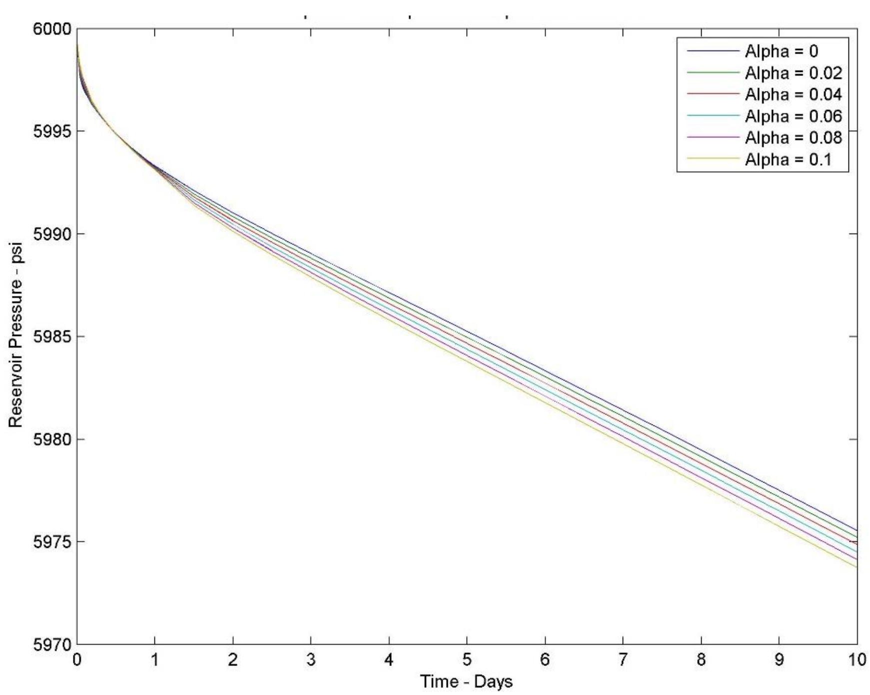

Figure 5-9 Grid Pressure of well no.3 change with alpha

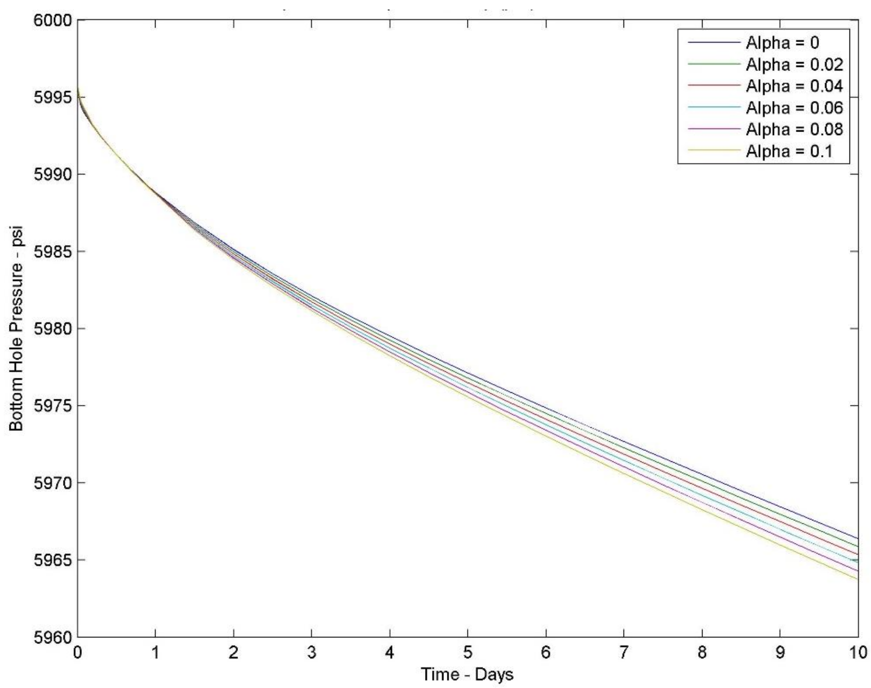

Figure 5-10 Pwf of well no.1 change with alpha

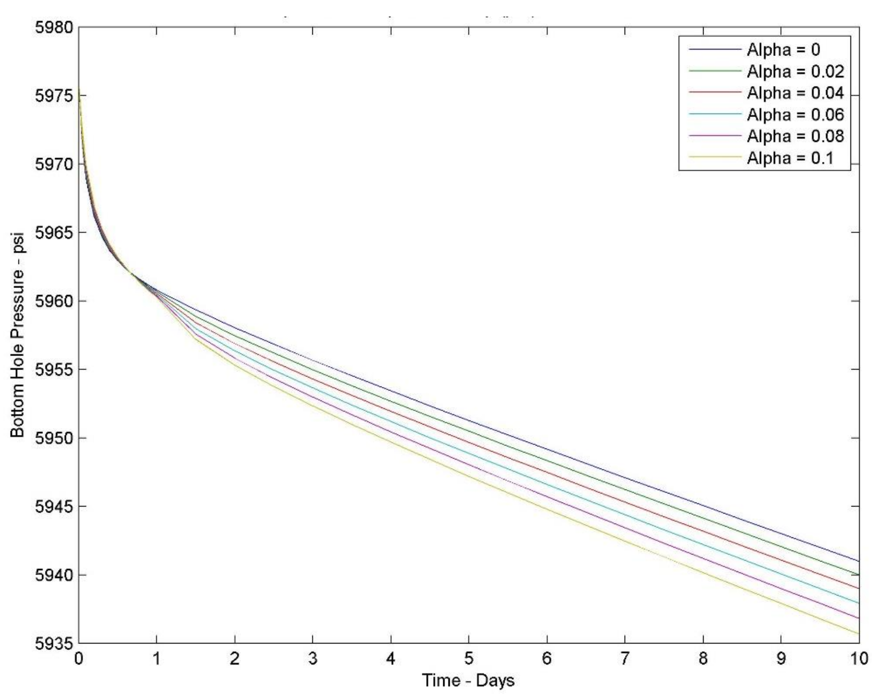

Figure 5-11 Pwf of well no.2 change with alpha

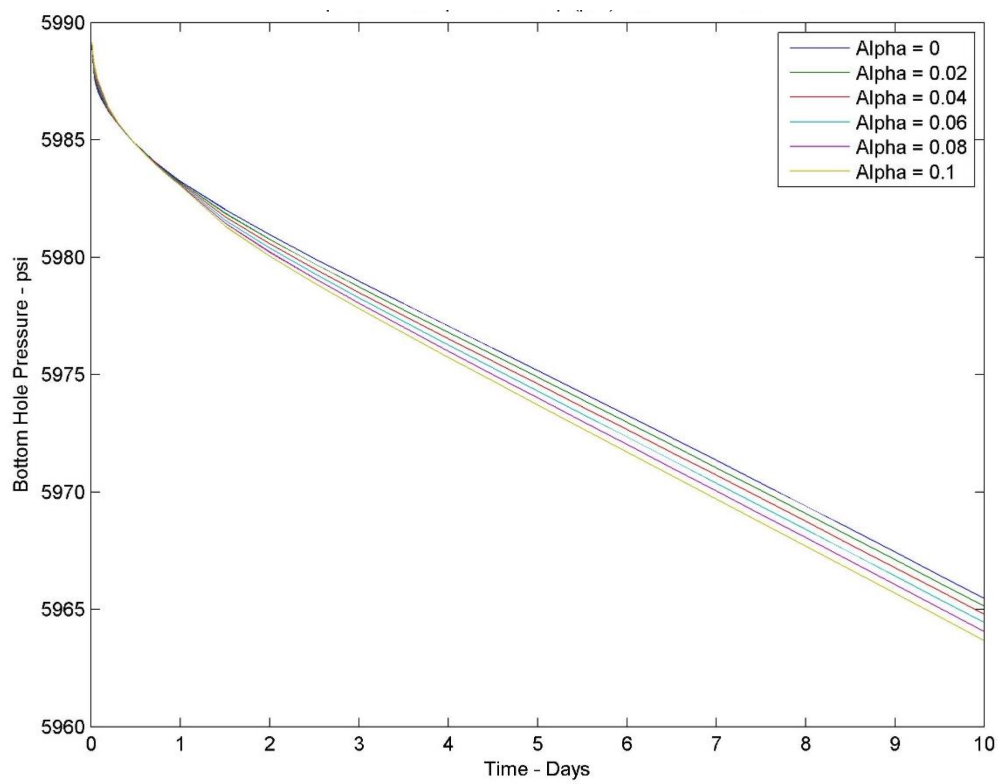

Figure 5-12 Pwf of well no.3 change with alpha

From the previous figures (Figure 5-7 to Figure 5-12), it is clear that the pressure drops in the wells drops with the increase of alpha (α) value as expected. Also we can see that the model matches the original model (with no alpha) at the value of zero which confirms the programming validity to certain extent.

The importance of considering the long memory effect and the value of alpha is clear from this simple sensitivity study, neglecting the effect while it exists will lead to non-accurate results and bad estimations, this is clearly shown in the inverse model illustrated in the next section.

5.3 Parameters Estimation - Levenberg-Marquardt Algorithm

Reservoir simulation is mainly used to predict the behavior of the reservoir and well bore pressure by supplying the reservoir properties to the mathematical model.

For our analysis, a mathematical model is used that is supposed to give the same output pressure response of the actual reservoir system (in case the memory effect was present). We will use a method called inverse analysis to see the effect of the memory presence on the reservoir parameters estimation and response matching.

Each reservoir system performs differently so a unique mathematical model is required for every reservoir system. However, due to limitations in modeling the diffusive nature of the pressure response in simulation, only a fixed number of mathematical models are available for studying the reservoir system. Here, we present a mathematical model that mimics a turbulent gas reservoir behavior while taking the memory effect into consideration.

Usually, the measured pressure data (the actual pressure response from the field) cannot be the same as the pressure response computed using a mathematical model because of the measurement errors and the simplified nature of model (Watson et al., 1988). Nowadays measurement errors are greatly reduced by the use of advance electronic gauges that give accurate pressure measurements. On the other hand, modeling errors is always present in the analysis due to simplicity and assumptions considered in development of a mathematical model. So there’s usually acceptable error and difference in readings during this process.

In this section we'll generate a dataset representing the actual reservoir response, and try to match these data using our mathematical model solving the inverse problem. Some errors and no uniqueness due to inverse nature of the problem is inherited. However, the final solution of the inverse problem is examined to give the minimum error between measured pressure response and model pressure, showing the effect of the memory effect.

Nonlinear regression technique is being used in modern Reservoir simulators. This technique became the standard industry practice in early 90's after the era of type curves. Nonlinear regression is also known as automated type curve matching. In this technique, the objective is to minimize the sum of squares of the difference between the observed pressure data and the model pressures. However this technique has disadvantage of getting trapped in local minima which is usually in the vicinity of initial guess.

In this section, we compute the reservoir and well bore parameters including the memory parameter for two synthetic cases using Levenberg-Marquardt Algorithm. All the examples involved single-phase flow, i.e. gas flow, in the reservoir with a constant production rate for each well. Wells are located at different locations in the reservoir, the actual values of the production rates of the wells and their locations are shown in Table 4 below:

Table 4 Wells Properties in Reservoir Model

Well #	Location (x,y,z)	Fixed Rate (MMscf)	State
1	3,3,1	10	Production
2	8,5,1	7	Production
3	15,13,1	8	Production

The objective function used in calculating various well and reservoir parameters is L2-norm. Evaluating the L2-Norm (also called as sum of error squares) for each possible solution requires calculating a model pressure. The results of model pressure are then used to compute the L2-Norm.

We try to match the total number of data points which is different for each case according the grid size. The algorithm is run for two different sets of parameters, in both cases the parameters are estimated assuming some initial guess of memory parameter permeability.

5.3.1 Example 1

In this example we use 16x16 grid system, the reservoir properties are the same as shown before, and the value of alpha that was used to generate the model to be match also shown in Table 5:

Table 5 Parameters used for reservoir formulation – Example 1

Parameter	**Value**
No. Grids in X direction	16
No. Grids in Y direction	16
No. Grids in Z direction	1
Length in X direction	6400 ft
Length in Y direction	4800 ft
Length in Z direction	60 ft
α	0.2
ϕ	0.15
Temperature	200^{o} F
S_g	0.65

P_i	6000 psi
P_g ref	6000 psi
B_g ref	187.62

The permeability distribution used for this run was ranging from

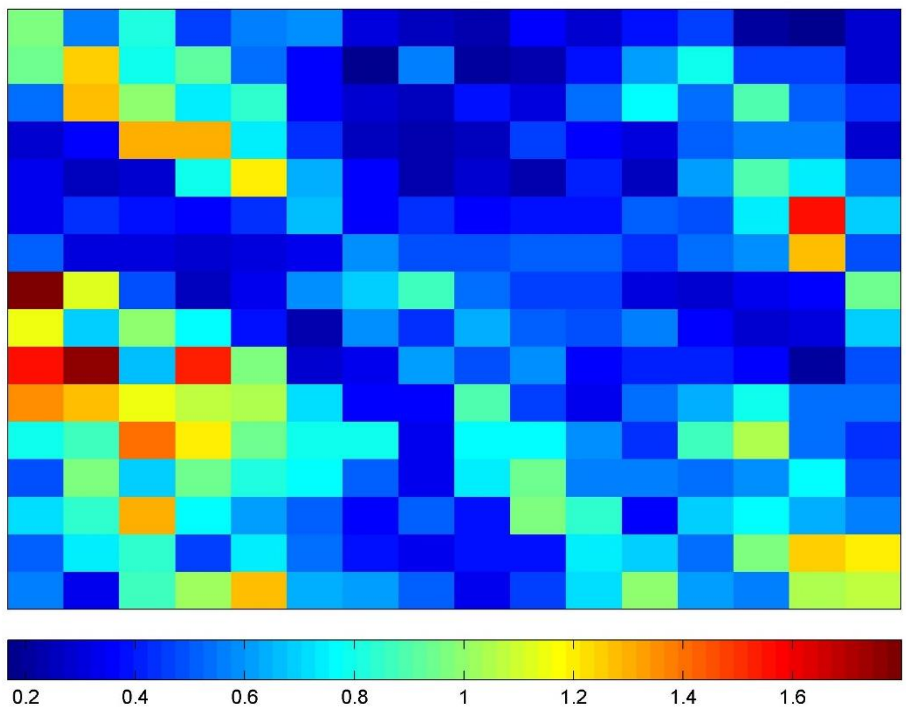

Figure 5-13 16x16 grid perm distribution

All these properties together generated a pressure response array that was used as the measured data array. We then tried to match that array by regenerating a pressure response using the proposed mathematical model.

The process was run four times, two times with synthetic data having the value of alpha = 0.2, and other two runs with the with synthetic data having the value of alpha = 0.

These runs are illustrated in the following two cases.

5.3.1.1 Case 1

The process was run twice in this case (one while considering anomalous flow, and the other without its consideration), and the results varied significantly as shown in Figure 5-14:

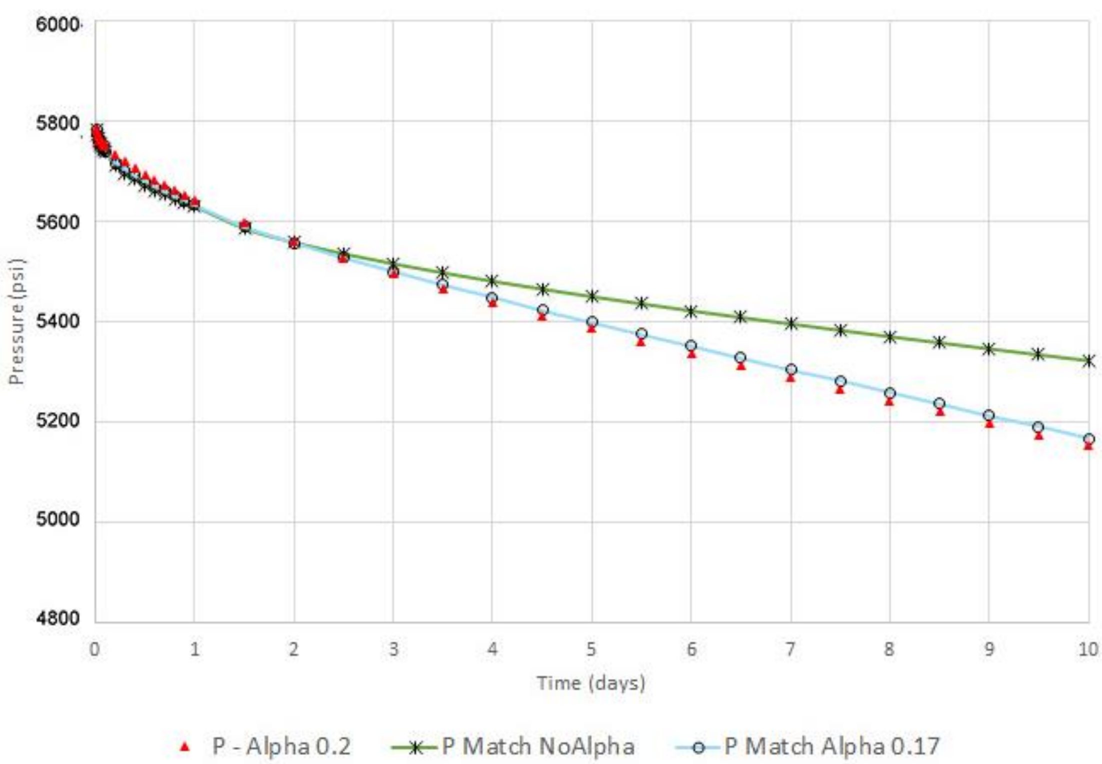

Figure 5-14 Example 1 – Case 1 - Parameters Estimation in Well 1

In Figure 5-14, the red dots show the actual reservoir pressure while the blue line indicates the matching with alpha, which is pretty good. As we can see, the match of non-alpha which is the green line, is very poor and it took double the supposed time for calculation as shown in Figure 5-16.

5.3.1.2 Case 2

In the second case, we tried to match synthetic that was generated without considering the anomalous flow, to see if our work trying to get alpha (while it is actually zero) would give worse match.

In Figure 5-15 we can see that although the match while not considering the anomalous flow is slightly better, the case in which alpha is being calculated is not bad at all. The results are very close and a low value of 0.09 was calculated for alpha which helped the better match even while the initial data has 0 value of alpha.

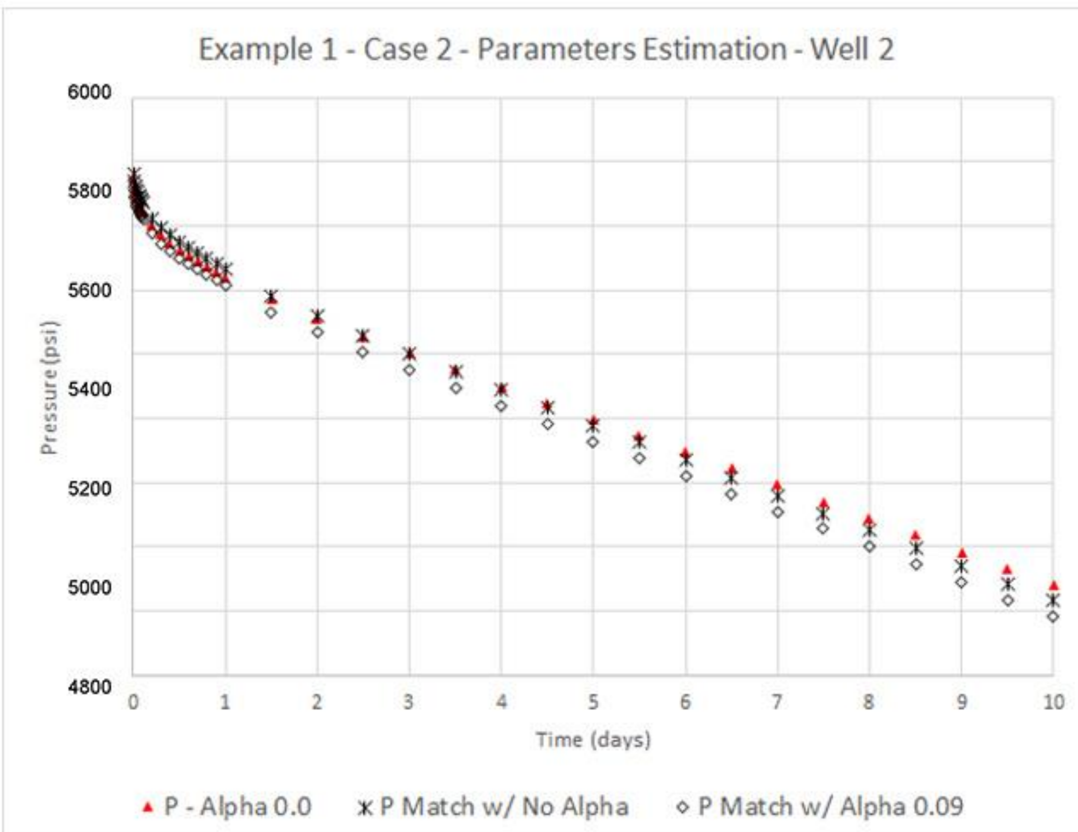

Figure 5-15 Example 1 – Case 2 - Parameters Estimation with no alpha initially in Well 2

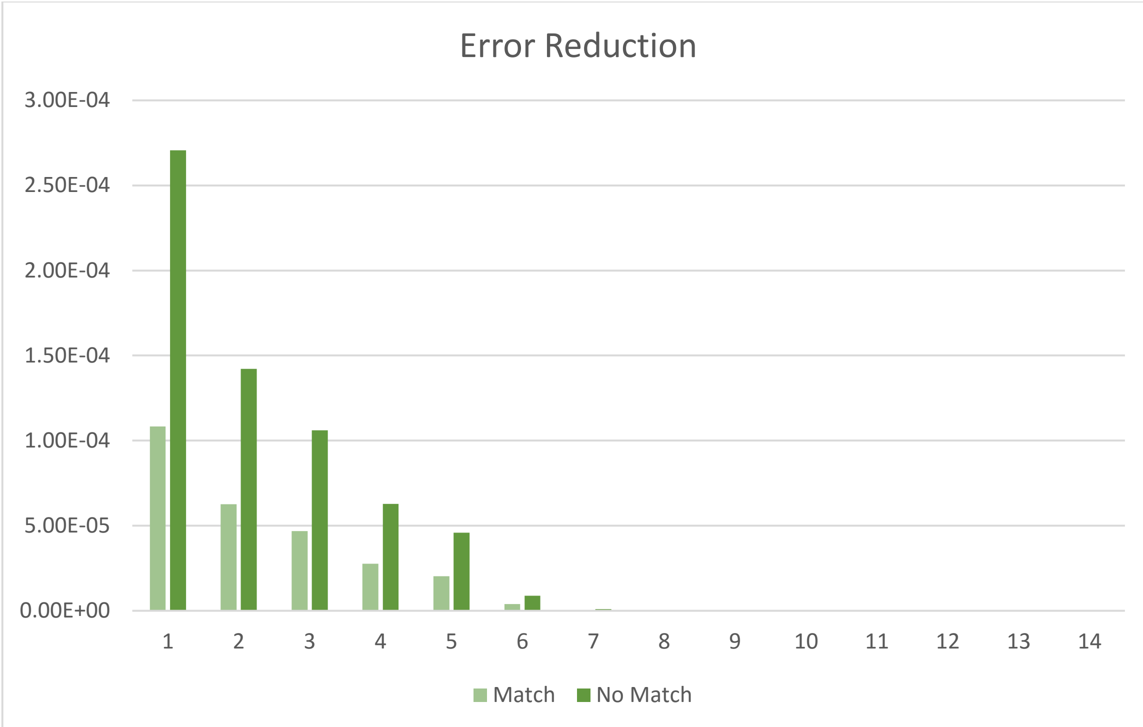

Figure 5-16 Error reduction in History Matching in Example 1

In Figure 5-16, we can see that the results of the match processes while considering the existence of alpha in case 1 is considered very good, originally the value of alpha was 0.2 and the match gave a value of 0.17 with greatly improved time and error reduction.

5.3.2 Example 2

In this example we use 48x48 grid system, the reservoir properties are the same as shown before, and the value of alpha that was used to generate the model to be match also shown in Table 6. Also two test cases were run exactly as in Example 1 but with the larger reservoir size. The results are almost identical as the low permeability case the same cells to be affected in both Examples.

Table 6 Parameters used for reservoir formulation - Example 2

Parameter	**Value**
No. Grids in X direction	48
No. Grids in Y direction	48
No. Grids in Z direction	1
Length in X direction	19200 ft
Length in Y direction	14400 ft
Length in Z direction	60 ft
α	0.2
ϕ	0.15
Temperature	200^{o} F
S_g	0.65
P_i	6000 psi
P_g ref	6000 psi
B_g ref	187.62

The permeability distribution used in the example is shown in Figure 5-17. The properties in Table 6 with the permeability distribution in Figure 5-17 generated a pressure response array that was used as the measured data array. We then tried to match that array by regenerating a pressure response using the proposed mathematical model. The process was repeated four times in two different cases, and the results varied significantly as shown in the following section.

5.3.2.1 Case 1

The process was run twice in this case (one while considering anomalous flow, and the other without its consideration), and the results varied significantly as shown in Figure 5-18

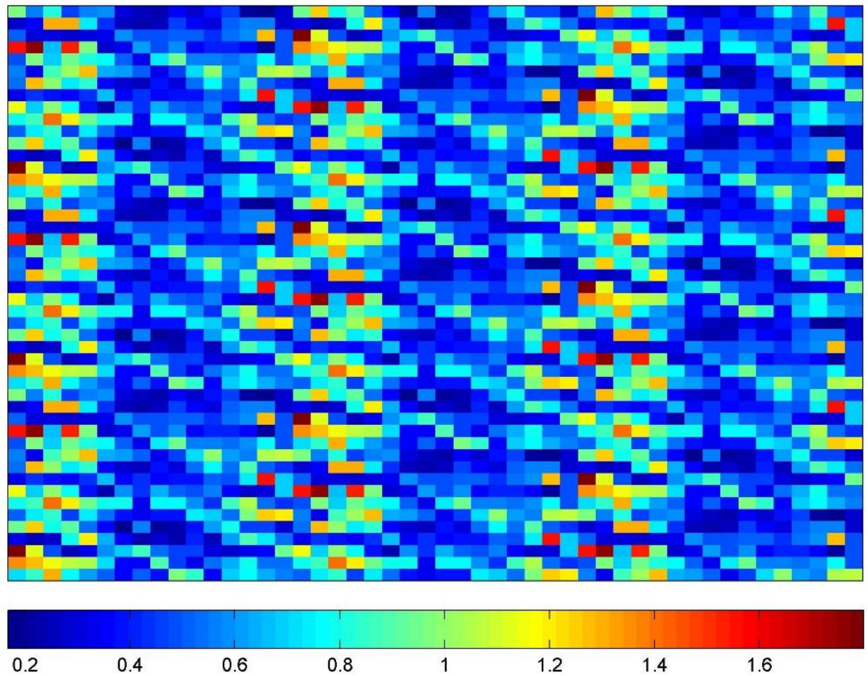

Figure 5-17 The permeability distribution used for this run 48x48

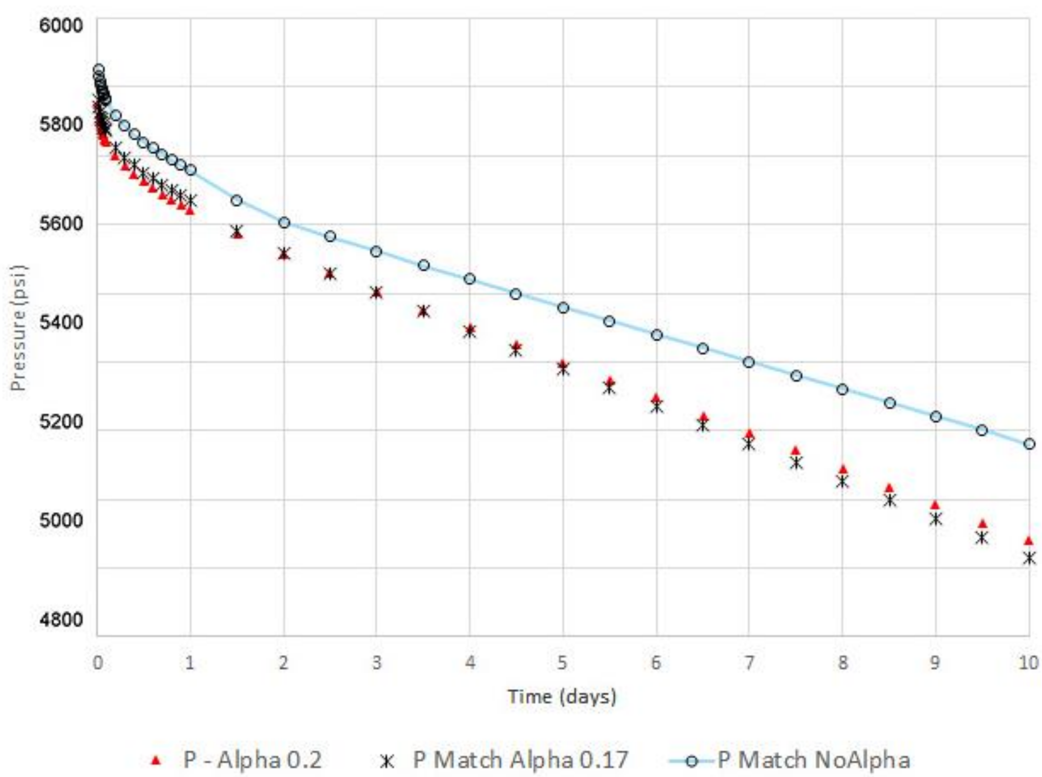

Figure 5-18 Example 2 - Parameters Estimation in Well 3

In Figure 5-18, the red dots show the actual reservoir pressure while the black dots indicate the matching with alpha, which is pretty good. As we can see, the match of non-alpha which is the green line, is very poor and it took double the supposed time for calculation as shown in the next graph.

5.3.2.2 Case 2

In the second case, we tried to match synthetic that was generated without considering the anomalous flow, to see if our work trying to get alpha (while it is actually zero) would give worse match.

In Figure 5-19 we can see the same pattern as in Example 1. The results are very close and a low value of 0.07 was calculated for alpha which helped the better match even while the initial data has 0 value of alpha.

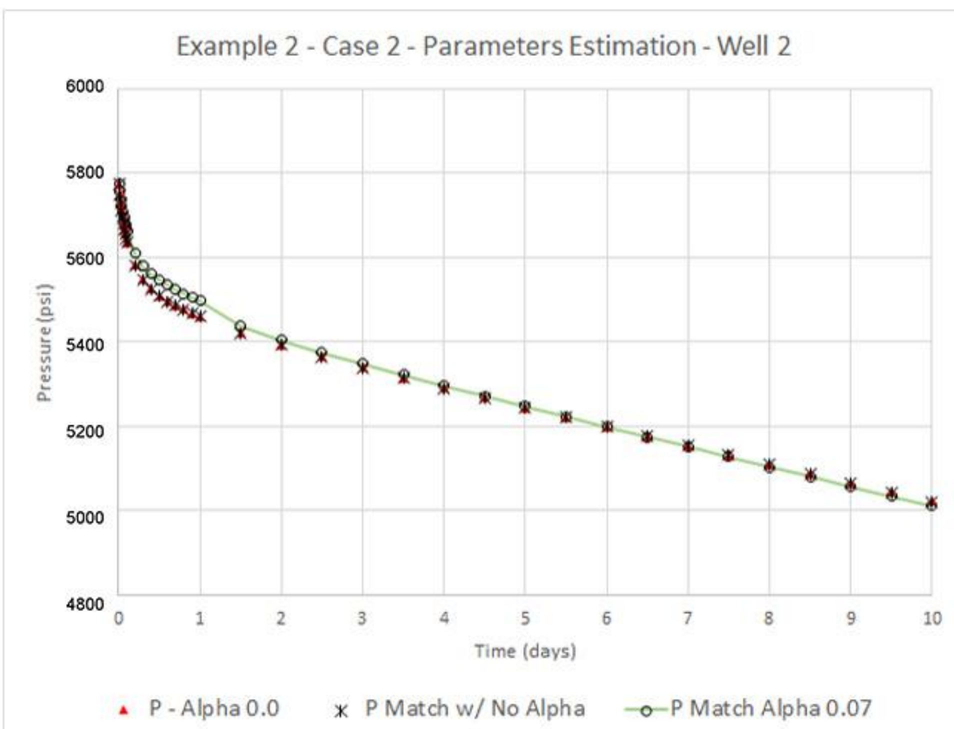

Figure 5-19 Example 2 – Case 2 - Parameters Estimation with no alpha initially in Well 2

The results of the match process while considering the existence of alpha is very interesting. Originally, the value of alpha was 0.2 and the match gave a value of 0.17 in case 1 with greatly improved time, and 0.07 in case 2 with little effect on the quality.

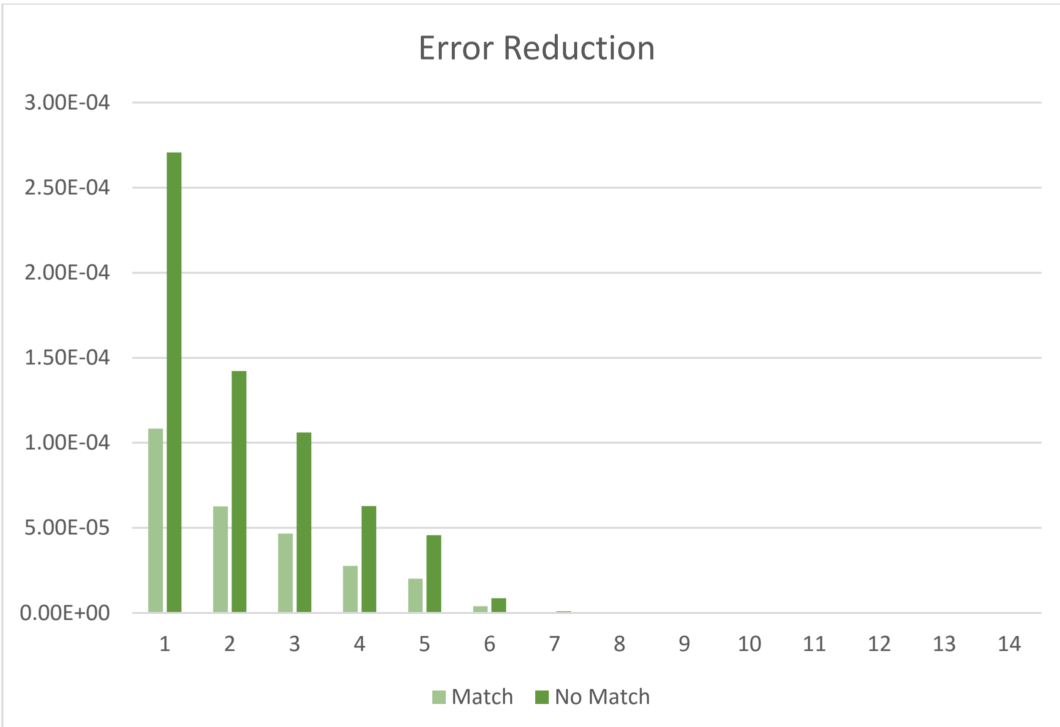

Figure 5-20 Error reduction in History Matching in Example 2

As shown in the previous section, the implementation of the memory formalism greatly enhanced the history matching process and gave much more accurate results.

CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS

It is believed that such effect is indeed present in most of the gas reservoirs, where turbulent flow is encountered. And thus, the consideration of the memory effect in the simulators used with these reservoirs, would lead to much better results and performance in the history matching processes.

Results show that the bottom hole pressure is affected by memory parameter and that the α presence will affect the calculation of permeability values from graphical analysis. Also, permeability and (α) are estimation using non-linear regression (Levenberg-Marquardt algorithm) considering both normal and fractional diffusion showed the importance of the model modification on the parameters estimation process.

6.1 Main Summary Points

- A modified model based on forchheimer's equation accounting for anomalous flow of turbulent gas has been derived
- The model has been solved numerically to predict the behavior of turbulent gas flow while taking the memory effect in account.
- Sensitivity study for various values of fractional order (alpha) has been carried out
- Sensitivity study for various grid block sizes on the same reservoir has been carried out

- Two examples of inverse modelling to estimate parameters, using LM method, has been examined
- In each example, two cases of matching (considering and not considering memory effect) are tested.

6.2 Main Observations

- Considering β in the gas calculations is significantly important, as expected
- Alpha has notable effect on the pressure drop. The higher the value of alpha, the higher the pressure drop as the turbulent effect becomes clearer
- At very early time , alpha has inverse effect on the pressure. That's is confirmed in the literature
- Grid size doesn't have an impact on the model as normally this would reflect the numerical dispersion effect which wouldn't be clear on the pressure model
- When estimating parameters without considering the memory effect, the matching time is significantly increased, while the matching accuracy decreased
- Some history mismatching cases in the industry could be due to neglecting the anomalous diffusion.

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